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Explorations of the extended ncKP hierarchy

Aristophanes Dimakis[1] **and Folkert Müller-Hoissen**[2]

[1] Department of Financial and Management Engineering, University of the Aegean, 31 Fostini Str., GR-82100 Chios, Greece

[2] Max-Planck-Institut für Strömungsforschung, Bunsenstrasse 10, D-37073 Göttingen, Germany

E-mail: dimakis@aegean.gr and fmuelle@gwdg.de

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Abstract

A recently obtained extension (xncKP) of the Moyal-deformed KP hierarchy (ncKP hierarchy) by a set of evolution equations in the Moyal-deformation parameters is further explored. Formulae are derived to compute these equations efficiently. Reductions of the xncKP hierarchy are treated, in particular to the extended ncKdV and ncBoussinesq hierarchies. Furthermore, a good part of the Sato formalism for the KP hierarchy is carried over to the generalized framework. In particular, the well-known bilinear identity theorem for the KP hierarchy, expressed in terms of the (formal) Baker–Akhiezer function, extends to the xncKP hierarchy. Moreover, it is demonstrated that N-soliton solutions of the ncKP equation are also solutions of the first few deformation equations. This is shown to be related to the existence of certain families of algebraic identities.

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1. Introduction

The *noncommutative KP hierarchy* (see [1], in particular) is defined as the set of equations

$$L_{t_n} := \frac{\partial L}{\partial t_n} = L^{(n)} * L - L * L^{(n)} =: [L^{(n)}, L]_* = [L, \bar{L}^{(n)}]_* \qquad n = 1, 2, \ldots \tag{1.1}$$

in terms of formal pseudo-differential operators

$$L = \partial + \sum_{k=1}^{\infty} u_{k+1} \partial^{-k} \qquad L^{(n)} = (L^n)_{\geqslant 0} \qquad \bar{L}^{(n)} = (L^n)_{<0} = L^n - L^{(n)} \tag{1.2}$$

with (matrices of) functions u_k and $L^n = L^{n-1} * L$. The non-negative (negative) part of a formal series is understood in the sense of non-negative (negative) powers of the operator ∂

of partial differentiation with respect to $x = t_1$. ∂ has to be a derivation of the associative product $*$. In this work, we assume that the product is given by

$$f * g = \mathbf{m} \circ \mathrm{e}^{P/2}(f \otimes g) \qquad P = \sum_{m,n=1}^{\infty} \theta_{m,n} \partial_{t_m} \otimes \partial_{t_n} \tag{1.3}$$

where $\mathbf{m}(f \otimes g) = fg$ for functions f, g, and $\theta_{n,m} = -\theta_{m,n}$ are constants. This implies that

$$(f * g)_{\theta_{m,n}} = f_{\theta_{m,n}} * g + f * g_{\theta_{m,n}} + \tfrac{1}{2}\big(f_{t_m} * g_{t_n} - f_{t_n} * g_{t_m}\big). \tag{1.4}$$

The *ncKP hierarchy* obtained in this way [2, 3] is thus a Moyal-deformation of the classical KP hierarchy (see [4–9], for example). Such deformations of soliton equations have recently been discussed in several papers (see [2, 3, 10, 11] and the references therein), partly motivated by the appearance of related structures in string theory.

It has been shown in [3] that the ncKP hierarchy can be extended to a bigger hierarchy, called the *xncKP hierarchy* in the following, by including the further (deformation) equations

$$L_{\theta_{m,n}} = [W^{(m,n)}, L]_* + \tfrac{1}{2}\big(L_{t_n} * L^{(m)} - L_{t_m} * L^{(n)}\big) \tag{1.5}$$

where

$$W^{(m,n)} = \tfrac{1}{2}(\bar{L}^{(n)} * L^{(m)} - \bar{L}^{(m)} * L^{(n)})_{\geqslant 0}. \tag{1.6}$$

The corresponding flows commute with those of the ncKP hierarchy. More precisely, for fixed natural numbers k, m, n, the $\theta_{m,n}$-flow commutes with the t_k-flow if (1.1) holds for k, m, n. Furthermore, the deformation flows also commute with each other if the associated ncKP hierarchy equations hold. To be more precise, if we fix four natural numbers k, l, m, n, the $\theta_{k,l}$-flow commutes with the $\theta_{m,n}$-flow if (1.1) holds for k, l, m, n. The extension of the ncKP hierarchy considered here may therefore be regarded as being of 'second order'. It is *not* a direct extension of the ncKP hierarchy in the sense in which each of its members extends the set of remaining equations.

The xncKP hierarchy equations are the integrability conditions of the linear system

$$L * \psi = \lambda\psi \qquad \psi_{t_n} = L^{(n)} * \psi \qquad \psi_{\theta_{m,n}} = W^{(m,n)} * \psi. \tag{1.7}$$

In the case of the 'commutative' KP hierarchy, a solution ψ of the first two of equations (1.7) is given by the so-called (formal) *Baker–Akhiezer function* of the KP hierarchy, which plays a crucial role in the Sato formalism [5–9]. In section 2 we introduce a Baker–Akhiezer function for the ncKP hierarchy which will be important in the subsequent sections.

The construction of the Baker–Akhiezer function involves a pseudo-differential operator with which L is written as a dressing of ∂. An important step in Sato theory is to express the hierarchy equations in terms of this operator [4–9]. In section 3 we derive a corresponding formulation of the xncKP hierarchy. Section 4 then proves, in particular, that the xncKP hierarchy equations can be cast into *bilinear identities*, which again generalizes a classical result (see [7, 8], for example).

Equations (1.1) determine the u_k, $k > 2$, in terms of u_2 (see appendix A). Introducing a potential ϕ via

$$u_2 = \phi_x \tag{1.8}$$

and taking the residue[3] of (1.1), results in a set of evolution equations for ϕ:[4]

$$\phi_{t_n} = \mathrm{res}(L^n) \qquad n = 1, 2, \ldots. \tag{1.9}$$

[3] The residue of a formal series is the coefficient of the ∂^{-1} term in the series. Note that it is irrelevant on which side of ∂^{-1} one reads off the coefficient.

[4] Here and in the following, possible constants of x-integration are set to zero. This can be substantiated with the assumption that the fields and their derivatives vanish as $|x| \to \infty$. Note that the first ($n = 1$) of equations (1.9) is an identity.

Expressions for the first few residues appearing on the right-hand side in terms of the u_k are given in appendix B. In section 5 we derive more convenient expressions for the above equations and, more generally, for those of the xncKP hierarchy. In particular, via the detour through the extension of the ncKP hierarchy, we find formulae for the ncKP equations, which reduce their computation to certain recursion relations. These results no longer refer to the extension and are actually not restricted to the special choice of product (1.3). Thus, Moyal-deformation and extension in the aforementioned sense may even lead to new insights into the classical hierarchies.

Section 6 presents some concrete examples of xncKP hierarchy equations. Here we concentrate on expressing them in the form $\phi_{t_n} = K_n, \phi_{\theta_{m,n}} = K_{m,n}$, where the right-hand sides are expressed solely in terms of the potential ϕ and its derivatives with respect to x and $y = t_2$, at the expense of having to allow for x-integrals also.

Applying reduction methods to integrable equations leads to other integrable equations. Section 7 tackles the question of what we obtain in this way from the new (deformation) equations. In particular, we consider the reduction of the xncKP hierarchy to the extended ncKdV and ncBoussinesq hierarchies.

Given any solution of the KP equation (or another member of its hierarchy), a deformation equation allows us to compute a corresponding (formal) solution of the ncKP equation which reduces to the former at vanishing deformation parameter. This is done by calculating iteratively higher derivatives of ϕ with respect to $\theta_{m,n}$ at $\theta_{m,n} = 0$ and writing down a formal Taylor series. Because of the commutativity of the flows, this yields indeed a (formal) solution of the ncKP equation for any given initial KP data[5]. In such an approach, there is hardly a chance to solve the corresponding equations to all orders in the respective deformation parameter. However, using a power series expansion in a new parameter ϵ, deformed soliton solutions of the ncKP equation were indeed obtained to all orders in ϵ [13]. In section 8 we extend this method to some of the deformation equations. We also refer to [14–16] for other methods and special solutions of the ncKP equation.

Finally, section 9 contains some concluding remarks. Some supplementary material has been separated from the main text as a series of appendices A–F, in order to achieve a better readability of the main text.

2. Baker–Akhiezer function and its adjoint

Let us write the Lax operator L as a 'dressing' of ∂,

$$L = X * \partial * X^{-1} \tag{2.1}$$

where X is an invertible (formal) pseudo-differential operator

$$X = 1 + \sum_{i=1}^{\infty} w_i(t, \theta)\partial^{-i} \tag{2.2}$$

with (matrices of) functions w_i.[6] An important step in the Sato formalism is to express the KP hierarchy in terms of X (see [7, 8, 17], for example). We prove this result for the ncKP hierarchy.

Theorem 2.1. *The ncKP hierarchy equations (1.1) are equivalent to*

$$X_{t_n} = -(X * \partial^n * X^{-1})_{<0} * X. \tag{2.3}$$

[5] See also [12] for a corresponding calculation in the case of the ncKdV equation.

[6] See appendix C for formulae expressing the w_i in terms of the coefficients u_k of L.

Proof. Using $(X^{-1})_{t_n} = -X^{-1} * X_{t_n} * X^{-1}$, differentiation of (2.1) leads to

$$L_{t_n} = X_{t_n} * \partial * X^{-1} - X * \partial * X^{-1} * X_{t_n} * X^{-1} = \left[X_{t_n} * X^{-1}, L\right]_*.$$

If (2.3) holds, using $L^n = X * \partial^n * X^{-1}$ it follows that (1.1) is satisfied. Let us now assume that (1.1) holds. Inserting (2.1) in (1.1), after some manipulations we obtain

$$\left[X^{-1} * \left(X_{t_n} + (X * \partial^n * X^{-1})_{<0} * X\right)\right]_x = 0$$

which implies that

$$X_{t_n} + (X * \partial^n * X^{-1})_{<0} * X = X * C_n$$

with $C_n = \sum_{i=1}^{\infty} C_{n,i}\partial^{-i}$ independent of x. Differentiation with respect to t_m and using $(\bar{L}^{(m)})_{t_n} - (\bar{L}^{(n)})_{t_m} = [\bar{L}^{(m)}, \bar{L}^{(n)}]_*$, which follows from (1.1) (see also [3]), leads to $(C_n)_{t_m} - (C_m)_{t_n} + [C_m, C_n]_* = 0$ and thus $C_n = C^{-1} * C_{t_n}$ with $C = 1 + \sum_{i=1}^{\infty} c_i\partial^{-i}$ independent of x. Hence

$$X_{t_n} + (X * \partial^n * X^{-1})_{<0} * X = X * C^{-1} * C_{t_n}$$

and a transformation $X \to X * C$ yields (2.3).[7] □

Let $\xi = \sum_{i=1}^{\infty} t_i\lambda^i$. Since this is linear in the variables t_i, we have $f(\xi) * g(\xi) = f(\xi)g(\xi)$. In particular, $\mathrm{e}^{\xi} * \mathrm{e}^{-\xi} = 1$. Furthermore,

$$(\mathrm{e}^{\xi})_{t_n} = \lambda^n\,\mathrm{e}^{\xi} \qquad \partial^j\,\mathrm{e}^{\xi} = \lambda^j\,\mathrm{e}^{\xi} \qquad \forall j \in \mathbb{Z} \tag{2.4}$$

since $t_1 = x$. We define the *Baker–Akhiezer function*[8]

$$\psi = X * \mathrm{e}^{\xi} = \left(1 + \sum_{i=1}^{\infty} w_i(t,\theta)\lambda^{-i}\right) * \mathrm{e}^{\xi} = \hat{w}(t,\theta,\lambda) * \mathrm{e}^{\xi}. \tag{2.5}$$

It follows that ψ satisfies the equations of the linear system of the ncKP hierarchy,

$$L * \psi = X * \partial * X^{-1} * X * \mathrm{e}^{\xi} = \lambda X * \mathrm{e}^{\xi} = \lambda\psi \tag{2.6}$$

$$\psi_{t_n} = X_{t_n} * \mathrm{e}^{\xi} + X * \lambda^n \mathrm{e}^{\xi} = -\bar{L}^{(n)} * \psi + L^n * \psi = L^{(n)} * \psi \tag{2.7}$$

where we made use of (2.3) in the form $X_{t_n} = -\bar{L}^{(n)} * X$.

Next we introduce an *involution*. For a function f, let

$$f(\theta)^\dagger = f(-\theta) \tag{2.8}$$

(suppressing unaffected further arguments). As a consequence,

$$(f * g)^\dagger = g^\dagger * f^\dagger \tag{2.9}$$

for any two functions f, g. This extends to pseudo-differential operators via[9]

$$(f\partial^j)^\dagger = (-\partial)^j f^\dagger \qquad \forall j \in \mathbb{Z} \tag{2.10}$$

as an involution,

$$(A * B)^\dagger = B^\dagger * A^\dagger. \tag{2.11}$$

[7] X is determined by L via (2.1) up to transformations $X \to X * C$, where $C = 1 + \sum_{i=1}^{\infty} c_i\partial^{-i}$ with c_i independent of x. For the commutative case a proof can be found in [7], p 342. This is easily adapted to the noncommutative case under consideration.

[8] Here the operator X is evaluated on e^{ξ}. The latter is thus not treated as an operator.

[9] The case of matrix coefficients is covered if † acts as above on all matrix components, and also takes the transpose of the respective matrix.

In particular, the adjoint of L is given by

$$L^{\dagger} = -(X^{-1})^{\dagger} * \partial * X^{\dagger}. \tag{2.12}$$

Furthermore, $1 = (X * X^{-1})^{\dagger} = (X^{-1})^{\dagger} * X^{\dagger}$ and thus $(X^{-1})^{\dagger} = (X^{\dagger})^{-1}$. Let us also define an *adjoint* of the Baker–Akhiezer function,

$$\psi^* = (X^{\dagger})^{-1} * \mathrm{e}^{-\xi} = \hat{w}^*(t, \theta, \lambda)^{\dagger} * \mathrm{e}^{-\xi} \tag{2.13}$$

where we wrote

$$X^{-1} = 1 + \sum_{n=1}^{\infty} \partial^{-n} w_n^{(*)} \qquad \hat{w}^* = 1 + \sum_{n=1}^{\infty} w_n^{(*)} \lambda^{-n} \tag{2.14}$$

with functions $w_n^{(*)}$. Using $(X^{-1})_{t_m} = X^{-1} * \bar{L}^{(n)}$, it follows that

$$L^{\dagger} * \psi^* = \lambda \psi^* \qquad \psi^*_{t_n} = -L^{(n)\dagger} * \psi^*. \tag{2.15}$$

It is often helpful to convert more generally (formal) pseudo-differential operators into (formal) series in the variable λ, as done in (2.5) and (2.13),

$$A * \mathrm{e}^{\lambda x} = \sum_j a_j * \partial^j \mathrm{e}^{\lambda x} = \sum_j \lambda^j a_j * \mathrm{e}^{\lambda x} \tag{2.16}$$

where $A = \sum_j a_j \partial^j$.

3. An alternative formulation of the xncKP hierarchy

In section 2 we derived an equivalent formulation of the ncKP hierarchy equations (1.1) in terms of the dressing operator X. The following result shows that a corresponding formulation also exists for the xncKP hierarchy.

Theorem 3.1. *If (1.1) (or equivalently (2.3), see theorem 2.1) holds, then (1.5) is equivalent to*

$$X_{\theta_{m,n}} = -\tfrac{1}{2}\big(X_{t_m} * \partial^n * X^{-1} - X_{t_n} * \partial^m * X^{-1}\big)_{<0} * X. \tag{3.1}$$

This in turn is then equivalent to

$$X_{\theta_{m,n}} = \tfrac{1}{2}(\bar{L}^{(m)} * L^n - \bar{L}^{(n)} * L^m)_{<0} * X. \tag{3.2}$$

Proof. Differentiation of $X * X^{-1} = 1$ with respect to $\theta_{m,n}$, using (1.4), leads to

$$X_{\theta_{m,n}} * X^{-1} + X * (X^{-1})_{\theta_{m,n}} + \tfrac{1}{2}\big(X_{t_m} * (X^{-1})_{t_n} - X_{t_n} * (X^{-1})_{t_m}\big) = 0$$

and thus, using (2.3),

$$X * (X^{-1})_{\theta_{m,n}} = -X_{\theta_{m,n}} * X^{-1} - \tfrac{1}{2}(\bar{L}^{(n)} * \bar{L}^{(m)} - \bar{L}^{(m)} * \bar{L}^{(n)}).$$

Differentiation of (2.1) yields

$$L_{\theta_{m,n}} = X_{\theta_{m,n}} * \partial * X^{-1} + X * \partial * (X^{-1})_{\theta_{m,n}} + \tfrac{1}{2}\big(X_{t_m} * \partial * (X^{-1})_{t_n} - X_{t_n} * \partial * (X^{-1})_{t_m}\big)$$

and, by use of our previous result, (2.1), (2.3) and (1.1) (which was already shown to be equivalent to (2.3)),

$$L_{\theta_{m,n}} = \big[X_{\theta_{m,n}} * X^{-1}, L\big] + \tfrac{1}{2}\big(L_{t_m} * \bar{L}^{(n)} - L_{t_n} * \bar{L}^{(m)}\big).$$

Let us assume that (3.1) holds. With the help of (2.3), this equation can be written in the form (3.2). Using the last formula, it is then easy to see that (1.5) is satisfied.

Now let us assume that (1.5) holds. Combining it with the above expression for $L_{\theta_{m,n}}$ leads to

$$\left[X_{\theta_{m,n}} * X^{-1} - W^{(m,n)}, L\right] + \tfrac{1}{2}\left(L_{t_m} * L^n - L_{t_n} * L^m\right) = 0.$$

With the help of $L_{t_n} = -[\bar{L}^{(n)}, L]$, we find

$$\left[X_{\theta_{m,n}} * X^{-1} - W^{(m,n)} + \tfrac{1}{2}(\bar{L}^{(n)} * L^m - \bar{L}^{(m)} * L^n), L\right] = 0$$

and, with the definition (1.6),

$$\left[X_{\theta_{m,n}} * X^{-1} + \tilde{W}^{(m,n)}, L\right] = 0$$

where

$$\tilde{W}^{(m,n)} = \tfrac{1}{2}(\bar{L}^{(n)} * L^m - \bar{L}^{(m)} * L^n)_{<0} = \bar{W}^{(m,n)} - \tfrac{1}{2}[\bar{L}^{(m)}, \bar{L}^{(n)}].$$

Multiplying by X^{-1} from the left and by X from the right, and using (2.1), leads to

$$\left[X^{-1} * (X_{\theta_{m,n}} + \tilde{W}^{(m,n)} * X)\right]_x = 0$$

and thus

$$X_{\theta_{m,n}} = -\tilde{W}^{(m,n)} * X + X * C_{m,n}$$

where $(C_{m,n})_x = 0$. Next we insert this expression in the integrability conditions $X_{\theta_{m,n}t_r} - X_{t_r\theta_{m,n}} = 0$ and use (2.3). With the help of (1.5), which in terms of $\tilde{W}$ reads

$$L_{\theta_{m,n}} = -[\tilde{W}^{(m,n)}, L] + \tfrac{1}{2}\left(L_{t_m} * \bar{L}^{(n)} - L_{t_n} * \bar{L}^{(m)}\right),$$

and the θ–t-integrability conditions

$$\tilde{W}^{(m,n)}_{t_r} - \bar{L}^{(r)}{}_{\theta_{m,n}} - [\tilde{W}^{(m,n)}, \bar{L}^{(r)}] = \tfrac{1}{2}\left(\bar{L}^{(r)}_{t_n} * \bar{L}^{(m)} - \bar{L}^{(r)}_{t_m} * \bar{L}^{(n)}\right)$$

we finally obtain $(C_{m,n})_{t_r} = 0$, i.e., the coefficients of $C_{m,n}$ are only allowed to depend on the $\theta_{k,1}$. Finally, we make use of the θ–θ-integrability conditions:

$$\begin{aligned}\tilde{W}^{(m,n)}_{\theta_{r,s}} - \tilde{W}^{(r,s)}_{\theta_{m,n}} - [\tilde{W}^{(m,n)}, \tilde{W}^{(r,s)}] - \tfrac{1}{2}\big(\tilde{W}^{(m,n)}_{t_r} * \bar{L}^{(s)} \\ - \tilde{W}^{(m,n)}_{t_s} * \bar{L}^{(r)} - \tilde{W}^{(r,s)}_{t_m} * \bar{L}^{(n)} + \tilde{W}^{(r,s)}_{t_n} * \bar{L}^{(m)}\big) = 0.\end{aligned}$$

With their help, $X_{\theta_{m,n}\theta_{r,s}} - X_{\theta_{r,s}\theta_{m,n}} = 0$ leads to

$$(C_{m,n})_{\theta_{r,s}} - (C_{r,s})_{\theta_{m,n}} - [C_{m,n}, C_{r,s}] = 0$$

which implies $C_{m,n} = C^{-1}C_{\theta_{m,n}}$ (without $*$ since C is independent of the t_r). Now

$$X_{\theta_{m,n}} = -\tilde{W}^{(m,n)} * X$$

is achieved with the gauge transformation $X \to XC$, which does not affect (2.3). □

As a consequence of this theorem and theorem 2.1, the xncKP hierarchy can be defined alternatively by (2.3) and (3.1). Using (3.1), it is easily verified that the Baker–Akhiezer function ψ, which was shown to satisfy the first two equations of the linear system (1.7), also satisfies the last equation of (1.7).

4. Bilinear identities

The classical KP hierarchy can be expressed equivalently in terms of so-called bilinear identities (see [7, 8], for example). In this section, we prove a corresponding result for the xncKP hierarchy. Furthermore, we recall the route towards the introduction of the τ-function of the KP hierarchy and discuss briefly problems one has to face in an attempt to find a generalization.

Let X be the dressing operator introduced in section 2. Differentiation of the identity $X^{-1} * X = 1$ with respect to $\theta_{m,n}$, using (1.4), (2.3) and (3.2), leads to

$$\begin{aligned}(X^{-1})_{\theta_{m,n}} &= \tfrac{1}{2} X^{-1} * (-(\bar{L}^{(m)} * L^n - \bar{L}^{(n)} * L^m)_{<0} + [\bar{L}^{(m)}, \bar{L}^{(n)}]_*) \\ &= \tfrac{1}{2} X^{-1} * (\bar{L}^{(n)} * L^{(m)} - \bar{L}^{(m)} * L^{(n)})_{<0} \end{aligned} \tag{4.1}$$

and thus

$$\begin{aligned}\psi^*_{\theta_{m,n}} &= ((X^{-1})^\dagger)_{\theta_{m,n}} * \mathrm{e}^{-\xi} - \tfrac{1}{2}(\bar{L}^{(m)}\lambda^n - \bar{L}^{(n)}\lambda^m)^\dagger * \psi^* \\ &= -((X^{-1})_{\theta_{m,n}})^\dagger * \mathrm{e}^{-\xi} + \tfrac{1}{2}(L^m * \bar{L}^{(n)} - L^n * \bar{L}^{(m)})^\dagger * \psi^* \\ &= -((X^{-1})_{\theta_{m,n}})^\dagger * \mathrm{e}^{-\xi} + \tfrac{1}{2}(\bar{L}^{(n)} * L^{(m)} - \bar{L}^{(m)} * L^{(n)} - [L^{(m)}, L^{(n)}]_*)^\dagger * \psi^* \\ &= W^{(m,n)\dagger} * \psi^* - \tfrac{1}{2}[L^{(m)}, L^{(n)}]_*^\dagger * \psi^* \end{aligned} \tag{4.2}$$

with the help of (2.15).

We define the *residue* res_λ of a formal series in λ as the coefficient of the λ^{-1} term[10]. Some useful relations are derived below. We follow the treatment of the commutative case in [7] (see also [8]).

Lemma 4.1. *Let $A = \sum_j a_j \partial^j$, $B = \sum_k b_k \partial^k$ be (formal) pseudo-differential operators. Then*

$$\mathrm{res}_\lambda((A * \mathrm{e}^{\lambda x}) * (B * \mathrm{e}^{-\lambda x})^\dagger) = \mathrm{res}(A * B^\dagger). \tag{4.3}$$

Proof. Evaluation of the left-hand side yields

$$\begin{aligned}\mathrm{res}_\lambda((A * \mathrm{e}^{\lambda x}) * (B * \mathrm{e}^{-\lambda x})^\dagger) &= \mathrm{res}_\lambda\left(\sum_{j,k}(-1)^k \lambda^{j+k} a_j * \mathrm{e}^{\lambda x} * \mathrm{e}^{-\lambda x} * b_k^\dagger\right) \\ &= \sum_k (-1)^k a_{-k-1} * b_k^\dagger.\end{aligned}$$

This equals the right-hand side since

$$\mathrm{res}(A * B^\dagger) = \mathrm{res}\left(\sum_j a_j * \partial^j \sum_k (-\partial)^k * b_k^\dagger\right) = \sum_k (-1)^k a_{-k-1} * b_k^\dagger$$

using $\mathrm{res}([\partial^j, f]) = 0$ for all $j \in \mathbb{Z}$ and functions $f(x)$. □

Theorem 4.1 (bilinear identities). *For all $i_1, \ldots, i_p \in \mathbb{N} \cup \{0\}$ and $j_{1,2}, \ldots, j_{m,n} \in \mathbb{N} \cup \{0\}$,*

$$\mathrm{res}_\lambda\left(\left(\partial_{t_1}^{i_1} \cdots \partial_{t_p}^{i_p} \partial_{\theta_{1,2}}^{j_{1,2}} \cdots \partial_{\theta_{m,n}}^{j_{m,n}} \psi\right) * \psi^\ddagger\right) = 0 \tag{4.4}$$

with $\psi^\ddagger = (\psi^*)^\dagger$ holds as a consequence of the xncKP hierarchy equations.

[10] One often finds the operation of taking the residue equivalently defined as $\oint \mathrm{d}\lambda/(2\pi \mathrm{i})$ with a contour around $|\lambda| = \infty$ [7].

Proof. Since $\psi_{t_k} = L^{(k)} * \psi$, $\psi_{\theta_{m,n}} = W^{(m,n)} * \psi$, and $L^{(k)}$, $W^{(m,n)}$ are polynomials in ∂, it is sufficient to prove the case $i_1 = i \geqslant 0, i_2 = \cdots = j_{m,n} = 0$. With the help of the preceding lemma we find that

$$\begin{aligned}\mathrm{res}_\lambda((\partial^i\psi) * \psi^\ddagger) &= \mathrm{res}_\lambda((\partial^i X * \mathrm{e}^\xi) * ((X^\dagger)^{-1} * \mathrm{e}^{-\xi})^\dagger) \\ &= \mathrm{res}_\lambda((\partial^i X * \mathrm{e}^{\lambda x}) * ((X^\dagger)^{-1} * \mathrm{e}^{-\lambda x})^\dagger) \\ &= \mathrm{res}(\partial^i X * X^{-1}) = \mathrm{res}(\partial^i) = 0.\end{aligned}$$

□

Next we prove the converse: the bilinear equations (4.4) imply the xncKP hierarchy equations.

Theorem 4.2. *Let*

$$\psi := X * \mathrm{e}^\xi \qquad \psi^\ddagger := (X^\star * \mathrm{e}^{-\xi})^\dagger$$

satisfy the bilinear equations (4.4), where

$$X := 1 + \sum_{i=1}^\infty w_i(t,\theta)\partial^{-i} \qquad X^\star := 1 + \sum_{i=1}^\infty w_i^\star(t,\theta)(-\partial)^{-i}$$

are formal series with functions w_i and $w_i^\star$. Then $X^\star = (X^\dagger)^{-1}$ and

$$X_{t_n} = -\bar{L}^{(n)} * X, \qquad X_{\theta_{mn}} = \tfrac{1}{2}(\bar{L}^{(m)} * L^n - \bar{L}^{(n)} * L^m)_{<0} * X$$

where $L = X * \partial * X^{-1}$. Hence ψ is the Baker–Akhiezer function of the (extended) ncKP hierarchy.

Proof. From the above definitions we obtain

$$\psi = \left(1 + \sum_{i=1}^\infty w_i\lambda^{-i}\right) * \mathrm{e}^\xi \qquad \psi^\ddagger = \mathrm{e}^{-\xi} * \left(1 + \sum_{i=1}^\infty (w^\star)_i^\dagger\lambda^{-i}\right).$$

Let us assume that $\mathrm{res}_\lambda((\partial^i\psi) * \psi^\ddagger) = 0$ for all $i \geqslant 0$. With the help of the above lemma this yields

$$\mathrm{res}(\partial^i X * (X^\star)^\dagger) = \mathrm{res}_\lambda((\partial^i X * \mathrm{e}^\xi) * (X^\star * \mathrm{e}^{-\xi})^\dagger) = \mathrm{res}_\lambda((\partial^i\psi) * \psi^\ddagger) = 0.$$

Since by construction $X * (X^\star)^\dagger = 1 + Y$ with $Y = Y_{<0}$, the last equation implies $\mathrm{res}(\partial^i Y) = 0$ for all $i \geqslant 0$ and thus $Y = 0$. It follows that $X^\star = (X^\dagger)^{-1}$.

The proof that the ncKP hierarchy equations $X_{t_n} = -\bar{L}^{(n)} * X$ hold, can be carried out in the same way as in the commutative case [7, 8]. With their help the additional equations of the extended hierarchy can be derived as follows. We find

$$\begin{aligned}X_{\theta_{m,n}} * \mathrm{e}^\xi &= (X * \mathrm{e}^\xi)_{\theta_{m,n}} - \tfrac{1}{2}\big(X_{t_m}\partial^n - X_{t_n}\partial^m\big) * \mathrm{e}^\xi \\ &= (X * \mathrm{e}^\xi)_{\theta_{m,n}} + \tfrac{1}{2}(\bar{L}^{(m)} * L^n - \bar{L}^{(n)} * L^m) * X * \mathrm{e}^\xi\end{aligned}$$

and thus

$$\left(X_{\theta_{m,n}} - \frac{1}{2}(\bar{L}^{(m)} * L^n - \bar{L}^{(n)} * L^m)_{<0} * X\right) * \mathrm{e}^\xi = \left(\frac{\partial}{\partial\theta_{m,n}} + \frac{1}{2}(\bar{L}^{(m)} * L^n - \bar{L}^{(n)} * L^m)_{\geqslant 0}\right) * X * \mathrm{e}^\xi.$$

By application of (4.4),

$$\begin{aligned}0 &= \mathrm{res}_\lambda\left(\left[\partial^i\left(\frac{\partial}{\partial\theta_{m,n}} + \frac{1}{2}(\bar{L}^{(m)}L^n - \bar{L}^{(n)}L^m)_{\geqslant 0}\right) * X * \mathrm{e}^\xi\right] * (X^\star * \mathrm{e}^{-\xi})^\dagger\right) \\ &= \mathrm{res}_\lambda\left(\left[\partial^i\left(X_{\theta_{m,n}} - \frac{1}{2}(\bar{L}^{(m)}L^n - \bar{L}^{(n)}L^m)_{<0} * X\right) * \mathrm{e}^\xi\right] * (X^\star * \mathrm{e}^{-\xi})^\dagger\right)\end{aligned}$$

so that, using the lemma,

$$\mathrm{res}\big(\partial^i(X_{\theta_{m,n}} - \tfrac{1}{2}(\bar{L}^{(m)}L^n - \bar{L}^{(n)}L^m)_{<0} * X) * X^{-1}\big) = 0$$

for all $i \geqslant 0$. This implies (3.2). □

We have shown that the bilinear identities (4.4) are equivalent to the xncKP hierarchy equations[11]. They are generated by formal Taylor expansion of

$$\mathrm{res}_\lambda(\psi(t+a, \theta+\alpha, \lambda) * \psi^\ddagger(t, \theta, \lambda)) = 0 \tag{4.5}$$

where $t+a$ and $\theta+\alpha$ stand for the collection of $t_n + a_n$, respectively $\theta_{m,n} + \alpha_{m,n}$, with arbitrary constants a_n and $\alpha_{n,m} = -\alpha_{m,n}$. With the shift $t \to t-a$, this becomes

$$\mathrm{res}_\lambda(\psi(t, \theta+\alpha, \lambda) * \psi^\ddagger(t-a, \theta, \lambda)) = 0. \tag{4.6}$$

A similar shift in θ affects the $*$-product. To make this manifest, let us write $*_\theta$ instead of $*$. Using (2.5), we find

$$\psi(t, \theta+\alpha, \lambda) = \hat{w}(t, \theta+\alpha, \lambda) *_{\theta+\alpha} \mathrm{e}^\xi = \hat{w}(t+\alpha(\lambda), \theta+\alpha, \lambda) *_\theta \mathrm{e}^\xi \tag{4.7}$$

with

$$\alpha_m(\lambda) = \frac{1}{2}\sum_{n=1}^\infty \alpha_{m,n}\lambda^n. \tag{4.8}$$

Here we used the definition of the $*$-product with $P = P_\theta + P_\alpha$ and

$$P_\alpha w(t, \theta, \lambda) \otimes \mathrm{e}^\xi = 2\left(\sum_{m=1}^\infty \alpha_m(\lambda) w(t, \theta, \lambda)_{t_m}\right) \otimes \mathrm{e}^\xi \tag{4.9}$$

which implies that

$$\mathrm{e}^{P_\alpha/2} w(t, \theta, \lambda) \otimes \mathrm{e}^\xi = \left(\exp\left(\sum_{m=1}^\infty \alpha_m(\lambda)\partial_{t_m}\right) w(t, \theta, \lambda)\right) \otimes \mathrm{e}^\xi = w(t+\alpha(\lambda), \theta, \lambda) \otimes \mathrm{e}^\xi. \tag{4.10}$$

Using (2.5) and (2.13), equation (4.6) can now be turned into

$$\mathrm{res}_\lambda(\hat{w}(t+\alpha(\lambda), \theta+\alpha, \lambda) * \hat{w}^*(t-a, \theta, \lambda)\, \mathrm{e}^{-\xi(-a,\lambda)}) = 0 \tag{4.11}$$

where $\xi(-a, \lambda)$ is obtained from ξ by replacing t_n with $-a_n$ for all $n \geqslant 1$.

Choosing $a = [\lambda_1^{-1}] = \left(\frac{1}{\lambda_1}, \frac{1}{2\lambda_1^2}, \frac{1}{3\lambda_1^3}, \ldots\right)$, we find

$$\mathrm{e}^{-\xi(-[\lambda_1^{-1}],\lambda)} = \frac{1}{1-\lambda/\lambda_1} \tag{4.12}$$

and, with the help of the residue identity (see [8], for example),

$$\mathrm{res}_\lambda\left(\frac{f(\lambda)}{1-\lambda/\lambda'}\right) = \lambda' f(\lambda')_{<0} \tag{4.13}$$

for a formal series $f(\lambda) = \sum_{i=-\infty}^\infty f_i\lambda^{-i}$. Here $f(\lambda')_{<0}$ denotes the part of $f(\lambda)$ which only contains negative powers of λ, with λ replaced by λ'. With our special choice of a, (4.11) becomes

$$\big(\hat{w}(t+\alpha(\lambda_1), \theta+\alpha, \lambda_1) * \hat{w}^*\big(t-[\lambda_1^{-1}], \theta, \lambda_1\big)\big)_{<0} = 0. \tag{4.14}$$

[11] We refer to appendix C for a concrete evaluation of bilinear identities.

Alternatively, choosing $a = [\lambda_1^{-1}] + [\lambda_2^{-1}]$, we find

$$\mathrm{res}_\lambda(f(\lambda)\,\mathrm{e}^{-\xi(-a,\lambda)}) = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}[f(\lambda_1)_{<0} - f(\lambda_2)_{<0}] \tag{4.15}$$

by partial fraction decomposition. Now (4.11) yields

$$\begin{aligned}&\big(\hat{w}(t+\alpha(\lambda),\theta+\alpha,\lambda) * \hat{w}^*\big(t-[\lambda_1^{-1}]-[\lambda_2^{-1}],\theta,\lambda\big)\big)_{<0}\big|_{\lambda=\lambda_1}\\ &\qquad = \big(\hat{w}(t+\alpha(\lambda),\theta+\alpha,\lambda) * \hat{w}^*\big(t-[\lambda_1^{-1}]-[\lambda_2^{-1}],\theta,\lambda\big)\big)_{<0}\big|_{\lambda=\lambda_2}.\end{aligned} \tag{4.16}$$

In the non-deformed commutative case, we have $(\hat{w}(t,\lambda)\hat{w}^*(t-a,\lambda))_{<0} = \hat{w}(t,\lambda)\hat{w}^*(t-a,\lambda) - 1$, so that equations (4.14) and (4.16) combine to

$$\frac{\hat{w}(t,\lambda_1)}{\hat{w}\big(t-[\lambda_2^{-1}],\lambda_1\big)} = \frac{\hat{w}(t,\lambda_2)}{\hat{w}\big(t-[\lambda_1^{-1}],\lambda_2\big)} \tag{4.17}$$

which is solved by $\hat{w}(t,\lambda) = \tau(t-[\lambda^{-1}])/\tau(t)$ with a function τ (see also [7, 8], for example). In the noncommutative case, however, this does not work. We have seen, however, that many related results indeed generalize to the noncommutative setting.

5. Explicit formulae for the xncKP hierarchy equations

In this section, we derive explicit expressions for the equations of the xncKP hierarchy in terms of ϕ and its derivatives. A crucial role in the derivation is played by the deformation equations (1.5), which imply

$$\phi_{\theta_{m,n}} = \tfrac{1}{2}\,\mathrm{res}(\bar{L}^{(n)} * L^{(m)} - \bar{L}^{(m)} * L^{(n)}) \tag{5.1}$$

(see also [3]). This can be further elaborated as follows:

$$\begin{aligned}2\phi_{\theta_{m,n}} &= \mathrm{res}(\bar{L}^{(n)} * (L^m - \bar{L}^{(m)}) - (L^m - L^{(m)}) * L^{(n)})\\ &= \mathrm{res}(\bar{L}^{(n)} * L^m - L^m * L^{(n)})\\ &= \mathrm{res}(\bar{L}^{(n)} * L^m - L^{m+n} + L^m * \bar{L}^{(n)})\\ &= \mathrm{res}(-L^{m+n} - [\bar{L}^{(n)}, L^m]_* + 2\bar{L}^{(n)} * L^m)\\ &= -\phi_{t_{m+n}} + \phi_{t_m t_n} + 2\,\mathrm{res}(\bar{L}^{(n)} * L^m).\end{aligned} \tag{5.2}$$

Writing

$$\bar{L}^{(n)} = -\sum_{m=1}^{\infty} \sigma_m^{(n)} * L^{-m} \tag{5.3}$$

with (matrices of) functions $\sigma_m^{(n)}$ (see also [1, 18–21]), we obtain[12]

$$\begin{aligned}\phi_{\theta_{m,n}} &= -\frac{1}{2}\big(\phi_{t_{m+n}} - \phi_{t_m t_n}\big) - \sum_{i=1}^{\infty}\sigma_i^{(n)} * \mathrm{res}(L^{m-i})\\ &= -\frac{1}{2}\big(\phi_{t_{m+n}} - \phi_{t_m t_n}\big) - \sigma_{m+1}^{(n)} - \sum_{i=1}^{m-1}\sigma_{m-i}^{(n)} * \phi_{t_i}\end{aligned} \tag{5.4}$$

since $\mathrm{res}(L^{-1}) = 1$ and $\mathrm{res}(L^{-m}) = 0$ for $m > 1$ (see also appendix D). In particular, we have

$$\sigma_1^{(n)} = -\mathrm{res}(L^n) = -\phi_{t_n}. \tag{5.5}$$

[12] Whereas the left-hand side is antisymmetric in m, n, this is not manifest for the right-hand side. The symmetric part vanishes as a consequence of the recursion relations which are derived in the following.

Lemma 5.1. *The* $\sigma_m^{(n)}$, $n > 1$, *are determined via*

$$\sigma_m^{(n+1)} = \sigma_{m,x}^{(n)} + \sigma_{m+1}^{(n)} + \sigma_{n+m}^{(1)} - \sum_{j=1}^{n-1} \sigma_j^{(1)} * \sigma_m^{(n-j)} + \sum_{j=1}^{m-1} \sigma_{m-j}^{(n)} * \sigma_j^{(1)} \tag{5.6}$$

iteratively in terms of the $\sigma_k^{(1)}$.[13]

Proof. Using (5.3) on the right-hand side of $L^{n+1} = L * L^n$, and taking the non-negative part, we find

$$L^{(n+1)} = \partial L^{(n)} - \sigma_1^{(n)} - \sum_{m=1}^{n} \sigma_m^{(1)} * L^{(n-m)}$$

(see also [21], for example). Applying both sides of this equation to ψ, using (5.3) and the first two equations of the linear system (1.7), we obtain the above recursion formula. $\square$

Lemma 5.2. *The* $\sigma_m^{(n)}$, $n > 1$, *are iteratively determined by the formulae*

$$\sigma_m^{(n+1)} = \sigma_{m,t_n}^{(1)} + \sigma_{m+1}^{(n)} + \sigma_{n+m}^{(1)} - \sum_{j=1}^{n-1} \sigma_j^{(1)} * \sigma_m^{(n-j)} + \sum_{j=1}^{m-1} \sigma_j^{(1)} * \sigma_{m-j}^{(n)} \tag{5.7}$$

$$\sigma_m^{(1)} = -\frac{1}{m}\left(\phi_{t_m} + \sum_{j=1}^{m-1} \sigma_{m-j,t_j}^{(1)}\right). \tag{5.8}$$

Proof. Applying (5.3) to ψ and using the linear system (1.7), we obtain

$$\psi_{t_n} = \left(\lambda^n + \sum_{m=1}^{\infty} \sigma_m^{(n)} \lambda^{-m}\right)\psi. \tag{5.9}$$

The integrability conditions $\psi_{t_m t_n} = \psi_{t_n t_m}$ lead to $\sigma_{1,t_n}^{(m)} = \sigma_{1,t_m}^{(n)}$ and

$$\sigma_{i,t_n}^{(m)} - \sigma_{i,t_m}^{(n)} + \sum_{j=1}^{i-1} \left[\sigma_{i-j}^{(m)}, \sigma_j^{(n)}\right]_* = 0 \qquad i = 2, 3, \ldots.$$

For $n = 1$, this reads[14]

$$\sigma_{i,x}^{(m)} = \sigma_{i,t_m}^{(1)} + \sum_{j=1}^{i-1} \left[\sigma_j^{(1)}, \sigma_{i-j}^{(m)}\right]_*.$$

Combining this equation with (5.6) leads to (5.7). Setting $i = n - m$ and summing over m (from 1 to $n-1$) yields

$$\sigma_1^{(n)} - n\sigma_n^{(1)} = \sum_{m=1}^{n-1} \sigma_{n-m,t_m}^{(1)}.$$

With the help of (1.9) and (5.5), this becomes (5.8). $\square$

[13] Lemma 5.3 determines the coefficients $\sigma_m^{(1)}$. See also appendix D for an alternative way of computing these functions.

[14] See also theorem 8.1.9 in [1]. As a consequence, in the commutative (undeformed) case, the functions $\sigma_m^{(1)}$ are conserved densities of the KP hierarchy [18, 19]. Kupershmidt ([1], p 128) suggests a notion of 'non-Abelian conserved density' as an expression, the t-derivative of which can be written as a sum of total x-derivative and commutators. In this sense, the $\sigma_m^{(1)}$ are common conserved densities of the ncKP hierarchy.

Lemma 5.3.

$$\sigma_n^{(1)} = p_n(-\tilde{\partial})\phi \tag{5.10}$$

with $\tilde{\partial} = \left(\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \ldots\right)$ *and the Schur polynomials*[15]

$$p_n(t) = \sum_{\substack{m_1+2m_2+\cdots+nm_n=n \\ m_i \geqslant 0}} \frac{1}{m_1! \cdots m_n!} t_1^{m_1} \cdots t_n^{m_n} \tag{5.11}$$

where $t = (t_1, t_2, \ldots)$.

Proof. The linear recursion formula (5.8) does not feel the non-commutativity. Hence, as in the commutative case, it leads to (5.10). See [20, 21], for example. □

Remark. The Schur polynomials satisfy

$$\mathrm{e}^{\xi(t,\lambda)} = \sum_{n=0}^{\infty} \lambda^n p_n(t). \tag{5.12}$$

As a consequence,

$$\sum_{n=1}^{\infty} \sigma_n^{(1)} \lambda^{-n} = \phi(t - [\lambda^{-1}]) - \phi(t) \tag{5.13}$$

with $\left[\lambda_1^{-1}\right] = \left(\frac{1}{\lambda_1}, \frac{1}{2\lambda_1^2}, \frac{1}{3\lambda_1^3}, \ldots\right)$. Differentiating (2.5) with respect to x and comparing the result with (5.9), we find $\hat{w}_x = (\phi(t - [\lambda^{-1}]) - \phi(t)) * \hat{w}$. In the commutative case, setting $\phi = (\ln\tau)_x$ then leads to $\hat{w} = \tau(t - [\lambda^{-1}])/\tau(t)$.

We can also evaluate (5.1) in the following alternative way:

$$\begin{aligned} 2\phi_{\theta_{m,n}} &= \mathrm{res}(L^m * \bar{L}^{(n)} - L^n * \bar{L}^{(m)}) \\ &= \mathrm{res}(L^m * \bar{L}^{(n)} - L^{m+n} + L^n * L^{(m)}) \\ &= \mathrm{res}\big(L^m * \bar{L}^{(n)} - L^{m+n} + L^{(m)} * L^n - (L^n)_{t_m}\big) \\ &= -\phi_{t_{m+n}} - \phi_{t_n t_m} + 2\,\mathrm{res}(L^m * \bar{L}^{(n)}). \end{aligned} \tag{5.14}$$

Instead of (5.3), this suggests to write

$$\bar{L}^{(n)} = \sum_{m=1}^{\infty} L^{-m} * \eta_m^{(n)} \tag{5.15}$$

with (matrices of) functions $\eta_m^{(n)}$. In particular, $\eta_1^{(n)} = \mathrm{res}(L^n) = \phi_{t_n}$.

Lemma 5.4. *The following recursion relations hold:*

$$\eta_m^{(n+1)} = -\eta_{m,t_n}^{(1)} + \eta_{m+1}^{(n)} + \eta_{n+m}^{(1)} - \sum_{j=1}^{m-1} \eta_{m-j}^{(n)} * \eta_j^{(1)} + \sum_{j=1}^{n-1} \eta_m^{(n-j)} * \eta_j^{(1)} \tag{5.16}$$

$$\eta_m^{(1)} = \frac{1}{m}\left(\phi_{t_m} + \sum_{j=1}^{m-1} \eta_{m-j,t_j}^{(1)}\right). \tag{5.17}$$

[15] The sum is over all partitions $(1^{m_1} 2^{m_2} \ldots n^{m_n})$ of n, such that $n = m_1 1 + m_2 2 + \cdots + m_n n$ with $m_i \in \mathbb{N} \cup \{0\}$. See [9], for example.

The last equation is solved by

$$\eta_n^{(1)} = p_n(\tilde{\partial})\phi. \tag{5.18}$$

Proof. Using the involution defined in section 2, (5.15) implies that

$$L^{(m)\dagger} = (L^\dagger)^m - \sum_{i=1}^{\infty} \eta_i^{(m)\dagger} * (L^\dagger)^{-i} \tag{5.19}$$

which applied to ψ^*, and with the help of (2.15), leads to

$$\psi^*_{t_m} = \left(-\lambda^m + \sum_{i=1}^{\infty} \eta_i^{(m)\dagger} \lambda^{-i}\right) * \psi^*. \tag{5.20}$$

The integrability conditions of these equations are

$$\eta_{i,t_n}^{(m)} - \eta_{i,t_m}^{(n)} - \sum_{j=1}^{i-1} \left[\eta_j^{(m)}, \eta_{i-j}^{(n)}\right]_* = 0 \tag{5.21}$$

which for $n = 1$ become

$$\eta_{i,x}^{(m)} = \eta_{i,t_m}^{(1)} + \sum_{j=1}^{i-1} \left[\eta_{i-j}^{(m)}, \eta_j^{(1)}\right]_*. \tag{5.22}$$

Using (5.19) on the right-hand side of the identity $(L^\dagger)^{m+1} = L^\dagger * (L^\dagger)^m$, leads to

$$(L^{m+1})^\dagger = -\partial (L^m)^\dagger + \sum_{i=1}^{\infty} \eta_i^{(1)\dagger} * (L^{m-i})^\dagger.$$

Taking the non-negative part results in[16]

$$L^{(m+1)\dagger} = -\partial L^{(m)\dagger} - \eta_1^{(m)\dagger} + \sum_{i=1}^{m} \eta_i^{(1)\dagger} * L^{(m-i)\dagger}.$$

Acting with the last expression on ψ^* and using (5.19) and (5.22), we obtain the recursion formula (5.16). Setting $i = n - m$ and summing over m (from 1 to $n - 1$) leads to

$$n\eta_n^{(1)} = \eta_1^{(n)} + \sum_{m=1}^{n-1} \eta_{n-m,t_m}^{(1)}$$

and thus (5.17). Equation (5.18) is an obvious analogue of lemma 5.3. □

The last lemma supplies us with explicit expressions for the $\eta_m^{(n)}$. In particular, we obtain

$$\begin{aligned}
&\eta_1^{(1)} = \phi_x \qquad \eta_2^{(1)} = \tfrac{1}{2}\phi_y + \tfrac{1}{2}\phi_{xx} \qquad \eta_3^{(1)} = \tfrac{1}{3}\phi_{t_3} + \tfrac{1}{2}\phi_{xy} + \tfrac{1}{6}\phi_{xxx} \\
&\eta_4^{(1)} = \tfrac{1}{4}\phi_{t_4} + \tfrac{1}{3}\phi_{xt_3} + \tfrac{1}{8}\phi_{yy} + \tfrac{1}{4}\phi_{xxy} + \tfrac{1}{24}\phi_{xxxx} \\
&\eta_5^{(1)} = \tfrac{1}{5}\phi_{t_5} + \tfrac{1}{4}\phi_{xt_4} + \tfrac{1}{6}\phi_{yt_3} + \tfrac{1}{6}\phi_{xxt_3} + \tfrac{1}{8}\phi_{xyy} + \tfrac{1}{12}\phi_{xxxy} + \tfrac{1}{120}\phi_{xxxxx} \\
&\eta_6^{(1)} = \tfrac{1}{6}\phi_{t_6} + \tfrac{1}{5}\phi_{xt_5} + \tfrac{1}{8}\phi_{xxt_4} + \tfrac{1}{8}\phi_{yt_4} + \tfrac{1}{18}\phi_{t_3t_3} + \tfrac{1}{6}\phi_{xyt_3} + \tfrac{1}{18}\phi_{xxxt_3} \\
&\qquad\quad + \tfrac{1}{48}\phi_{yyy} + \tfrac{1}{16}\phi_{xxyy} + \tfrac{1}{48}\phi_{xxxxy} + \tfrac{1}{720}\phi_{xxxxxx}.
\end{aligned} \tag{5.23}$$

[16] Note that $(A_{\geqslant 0})^\dagger = (A^\dagger)_{\geqslant 0}$ for every pseudo-differential operator A.

Remark. Using (5.12) and (5.18), we get

$$\sum_{i=1}^{\infty} \eta_i^{(1)} \lambda^{-i} = \phi(t+[\lambda^{-1}]) - \phi(t). \tag{5.24}$$

Differentiation of (2.13) with respect to x and comparison of the result with (5.20), leads to $\hat{w}^*_x = \hat{w}^* * (\phi(t+[\lambda^{-1}]) - \phi(t))$. In the commutative case, setting $\phi = (\ln\tau)_x$ yields the expression $\hat{w}^* = \tau(t+[\lambda^{-1}])/\tau(t)$.

There is an involution ω, which relates the σ- and the η-coefficients. It is defined by

$$\left(f_{t_n}\right)^\omega = -(f^\omega)_{t_n} \qquad \left(f_{\theta_{m,n}}\right)^\omega = -(f^\omega)_{\theta_{m,n}} \qquad (f*g)^\omega = -g^\omega * f^\omega \qquad \phi^\omega = \phi. \tag{5.25}$$

Since this involution maps the σ-recursion relations (see lemma 5.2) into the η-recursion relations (lemma 5.4), and vice versa, it follows that

$$\left(\eta_m^{(n)}\right)^\omega = \sigma_m^{(n)}. \tag{5.26}$$

For example, explicit expressions for the $\sigma_m^{(1)}$ are now obtained in a simple way by applying the involution ω to (5.23). The recursion formula (5.16) yields

$$\eta_2^{(n)} = \phi_{t_{n+1}} + \phi_{xt_n} - \eta_{n+1}^{(1)} - \sum_{i=1}^{n-1} \phi_{t_{n-i}} * \eta_i^{(1)} \tag{5.27}$$

which, by application of ω, leads to

$$\sigma_2^{(n)} = -\phi_{t_{n+1}} + \phi_{xt_n} - \sigma_{n+1}^{(1)} - \sum_{i=1}^{n-1} \sigma_i^{(1)} * \phi_{t_{n-i}}. \tag{5.28}$$

Inserting (5.15) in (5.14), we obtain

$$\begin{aligned}\phi_{\theta_{m,n}} &= -\frac{1}{2}\left(\phi_{t_{m+n}} + \phi_{t_n t_m}\right) + \sum_{i=1}^{\infty} \mathrm{res}(L^{m-i}) * \eta_i^{(n)} \\ &= -\frac{1}{2}\left(\phi_{t_{m+n}} + \phi_{t_n t_m}\right) + \eta_{m+1}^{(n)} + \sum_{i=1}^{m-1} \phi_{t_i} * \eta_{m-i}^{(n)}.\end{aligned} \tag{5.29}$$

Addition of (5.4) and (5.29) yields

$$\phi_{\theta_{m,n}} = -\frac{1}{2}\left(\phi_{t_{m+n}} + \sigma_{m+1}^{(n)} - \eta_{m+1}^{(n)} + \sum_{i=1}^{m-1}\left(\sigma_{m-i}^{(n)} * \phi_{t_i} - \phi_{t_i} * \eta_{m-i}^{(n)}\right)\right) \tag{5.30}$$

whereas subtraction results in

$$\phi_{t_m t_n} = \sigma_{m+1}^{(n)} + \eta_{m+1}^{(n)} + \sum_{i=1}^{m-1}\left(\sigma_{m-i}^{(n)} * \phi_{t_i} + \phi_{t_i} * \eta_{m-i}^{(n)}\right). \tag{5.31}$$

In particular, from these equations we obtain

$$\phi_{\theta_{1,n}} = -\tfrac{1}{2}\phi_{t_{n+1}} + \tfrac{1}{2}\left(\eta_2^{(n)} - \sigma_2^{(n)}\right) \tag{5.32}$$

$$\phi_{\theta_{2,n}} = -\tfrac{1}{2}\phi_{t_{n+2}} + \tfrac{1}{2}\left(\eta_3^{(n)} - \sigma_3^{(n)}\right) + \tfrac{1}{2}\left\{\phi_x, \phi_{t_n}\right\}_* \tag{5.33}$$

$$\phi_{\theta_{3,n}} = -\tfrac{1}{2}\phi_{t_{n+3}} + \tfrac{1}{2}\left(\eta_4^{(n)} - \sigma_4^{(n)}\right) + \tfrac{1}{2}\left\{\phi_y, \phi_{t_n}\right\}_* + \tfrac{1}{2}\left(\phi_x * \eta_2^{(n)} - \sigma_2^{(n)} * \phi_x\right) \tag{5.34}$$

where $\{\,,\,\}_*$ denotes the anti-commutator, and

$$\left(\phi_{t_n}\right)_x = \eta_2^{(n)} + \sigma_2^{(n)} \tag{5.35}$$

$$\left(\phi_{t_n}\right)_y = \eta_3^{(n)} + \sigma_3^{(n)} + \left[\phi_x, \phi_{t_n}\right]_*. \tag{5.36}$$

Equations (5.30)–(5.36) are main results of this section. Note that (5.31) and the last two equations no longer involve θ-derivatives, although they originated from the deformation equations.

Remark. By inspection of the recursion relations in lemma 5.2, one finds that

$$\sigma_{m+1}^{(n)} = -\frac{n}{m+n}\phi_{t_{m+n}} + \cdots \tag{5.37}$$

where the remaining terms only contain t_k-derivatives of ϕ with $k < m+n$. Using the ω-involution to obtain a corresponding expression for $\eta_{m+1}^{(n)}$, we see that the leading derivative term on the right-hand side of (5.30) is $\frac{1}{2}(n-m)\phi_{t_{m+n}}$. Furthermore, the terms with $\phi_{t_{m+n}}$ cancel each other in the combination $\eta_{m+1}^{(n)} + \sigma_{m+1}^{(n)}$. Note, however, that this expression contains a term proportional to $\phi_{t_{m+n-1}x}$. This means that (5.35) is not yet solved for $\phi_{t_n x}$. But for $n > 2$ it can indeed always be solved for $\phi_{t_n x}$ due to the following argument. Using (5.10), (5.18), and the combinatorial formula (5.11) for the Schur polynomials, it follows that

$$\sigma_{n+1}^{(1)} = -\frac{1}{n+1}\phi_{t_{n+1}} + \frac{1}{n}\phi_{xt_n} + \frac{1}{2(n-1)}\phi_{yt_{n-1}} - \frac{1}{2(n-1)}\phi_{xxt_{n-1}} + \cdots \tag{5.38}$$

$$\eta_{n+1}^{(1)} = \frac{1}{n+1}\phi_{t_{n+1}} + \frac{1}{n}\phi_{xt_n} + \frac{1}{2(n-1)}\phi_{yt_{n-1}} + \frac{1}{2(n-1)}\phi_{xxt_{n-1}} + \cdots \tag{5.39}$$

where the remaining terms only contain derivatives with respect to t_k with $k < n-1$. With the help of (5.27) and (5.28), (5.35) takes the form $\frac{1}{n}(n-2)\phi_{t_n x} + \cdots = 0$, where the dots stand for terms which contain t_k-derivatives of ϕ with $k < n$ only. For $n > 2$, we can thus solve for ϕ_{t_n} with an x-integration. In this way, one arrives iteratively at integro-differential expressions for the ϕ_{t_n}, which only contain derivatives with respect to x and y, and x-integrals. In section 6 we construct such a representation of the xncKP hierarchy equations in a different way.

With the help of the above results, we can further evaluate (5.32):

$$\phi_{\theta_{1,2}} = \tfrac{1}{6}\left(\phi_{t_3} - \phi_{xxx}\right) - \phi_x * \phi_x \tag{5.40}$$

$$\phi_{\theta_{1,3}} = \tfrac{1}{4}\left(\phi_{t_4} - \phi_{xxy} - 3\{\phi_x, \phi_y\}_* - [\phi_x, \phi_{xx}]_*\right) \tag{5.41}$$

$$\begin{aligned}\phi_{\theta_{1,4}} = {} & \tfrac{3}{10}\phi_{t_5} - \tfrac{1}{6}\phi_{xxt_3} - \tfrac{1}{8}\phi_{xyy} - \tfrac{1}{120}\phi_{xxxxx} - \tfrac{2}{3}\left\{\phi_x, \phi_{t_3}\right\}_* - \tfrac{1}{12}\{\phi_x, \phi_{xxx}\}_* \\ & - \tfrac{1}{4}[\phi_x, \phi_{xy}]_* + \tfrac{1}{4}[\phi_{xx}, \phi_y]_* - \tfrac{1}{2}\phi_y * \phi_y. \end{aligned} \tag{5.42}$$

For $n = 2$, (5.35) is identically satisfied. For $n = 3, 4, 5$, we obtain

$$\left(\phi_{t_3}\right)_x = \tfrac{1}{4}(3\phi_{yy} + \phi_{xxxx} - 6[\phi_x, \phi_y]_* + 6(\phi_x * \phi_x)_x) \tag{5.43}$$

$$\left(\phi_{t_4}\right)_x = \tfrac{1}{3}\left(2\phi_{yt_3} + \phi_{xxxy} - \left[\phi_x, 4\phi_{t_3} - \phi_{xxx}\right]_* + 3(\{\phi_x, \phi_y\}_*)_x\right) \tag{5.44}$$

$$\begin{aligned}\left(\phi_{t_5}\right)_x = {} & \tfrac{1}{216}\big(90\phi_{yt_4} + 40\phi_{t_3t_3} + 40\phi_{t_3xxx} + 45\phi_{xxyy} + \phi_{xxxxxx} + 270\left[\phi_{t_4}, \phi_x\right]_* \\ & + 60\left[\phi_{t_3}, \phi_y + 3\phi_{xx}\right]_* + 120\left\{\phi_{xt_3}, \phi_x\right\}_* + 45\{\phi_{yy}, \phi_x\}_* \\ & + 180\{\phi_{xy}, \phi_y\}_* + 60[\phi_y, \phi_{xxx}]_* + 90[\phi_x, \phi_{xxy}]_* + 15\{\phi_{xxxx}, \phi_x\}_*\big). \end{aligned} \tag{5.45}$$

Now let us elaborate (5.33) and (5.36). Setting $n = 1$ in (5.16) yields

$$\eta_n^{(2)} = -\eta_{n,x}^{(1)} + 2\eta_{n+1}^{(1)} - \sum_{i=1}^{n-1} \eta_i^{(1)} * \eta_{n-i}^{(1)} \tag{5.46}$$

and its ω-dual

$$\sigma_n^{(2)} = \sigma_{n,x}^{(1)} + 2\sigma_{n+1}^{(1)} + \sum_{i=1}^{n-1} \sigma_i^{(1)} * \sigma_{n-i}^{(1)}. \tag{5.47}$$

Furthermore, setting $m = 2$ in (5.16), leads to

$$\eta_3^{(n)} = \frac{1}{2}\left(\phi_{yt_n} + \phi_{xxt_n}\right) + \phi_{t_n} * \phi_x + \eta_2^{(n+1)} - \eta_{n+2}^{(1)} - \sum_{i=1}^{n-1} \eta_2^{(i)} * \eta_{n-i}^{(1)} \tag{5.48}$$

and a corresponding expression for $\sigma_3^{(n)}$ via ω-involution. By use of these expressions, (5.33) with $n = 3$ results in

$$\phi_{\theta_{2,3}} = \tfrac{1}{10}\phi_{t_5} - \tfrac{1}{8}\phi_{xyy} + \tfrac{1}{40}\phi_{xxxxx} - \tfrac{3}{4}\phi_y^{*2} - \tfrac{1}{4}[\phi_x, \phi_{xy}]_* + \tfrac{1}{4}\{\phi_x, \phi_{xxx}\}_* + \tfrac{1}{4}\phi_{xx}^{*2} + \phi_x^{*3} \tag{5.49}$$

where, for example, $\phi_y^{*2} = \phi_y * \phi_y$.

Equation (5.36) with $n = 3$ coincides with (5.44).

6. A special form of the xncKP equations

With the help of appendices A and B, we can evaluate the right-hand side of (1.9) directly in terms of ϕ and its derivatives with respect to x and $y = t_2$.[17] Introducing the abbreviations

$$\Phi^{(1)} := \phi_x \tag{6.1}$$

$$\Phi^{(2)} := \int \left(2[\Phi^{(1)}, \phi_x]_* + \Phi_y^{(1)}\right) \mathrm{d}x = \phi_y \tag{6.2}$$

$$\Phi^{(3)} := \int \left(2[\Phi^{(2)}, \phi_x]_* + \Phi_y^{(2)}\right) \mathrm{d}x = \int (2[\phi_y, \phi_x]_* + \phi_{yy})\, \mathrm{d}x \tag{6.3}$$

$$\Phi^{(4)} := \int \left(2[\Phi^{(3)}, \phi_x]_* + \Phi_y^{(3)}\right) \mathrm{d}x + 2\int \left\{\phi_x, \Phi_x^{(2)}\right\}_* \mathrm{d}x \tag{6.4}$$

$$\Phi^{(5)} := \int \left(2[\Phi^{(4)}, \phi_x]_* + \Phi_y^{(4)}\right) \mathrm{d}x + 2\int \left\{\phi_x, \Phi_x^{(3)}\right\}_* \mathrm{d}x + 2\int [\phi_{xx}, \phi_{xy}]_*\, \mathrm{d}x \tag{6.5}$$

we obtain the *(potential) ncKP equation*

$$\phi_{t_3} = \tfrac{1}{4}\left(\phi_{xxx} + 6\phi_x^{*2} + 3\Phi^{(3)}\right) \tag{6.6}$$

and the next two ($n = 4, 5$) evolution equations of the ncKP hierarchy:

$$\phi_{t_4} = \tfrac{1}{2}\phi_{xxy} + \{\phi_x, \phi_y\}_* + \tfrac{1}{2}\Phi^{(4)} \tag{6.7}$$

$$\phi_{t_5} = \tfrac{5}{8}\left(\tfrac{1}{10}\phi_{xxxxx} + \phi_{xyy} + 2\phi_y^{*2} + \{\phi_x, \phi_{xxx} + \Phi^{(3)}\}_* + \phi_{xx}^{*2} + 4\phi_x^{*3} + \tfrac{1}{2}\Phi^{(5)} - ([\phi_x, \phi_y]_*)_x\right). \tag{6.8}$$

[17] Alternatively, we can derive the resulting equations iteratively by solving equations like (5.43)–(5.45) for the (highest) t_n-derivative and eliminating t_k-derivatives with $2 < k < n$ on the right-hand sides of the resulting equations. The presentation in this section allows for a somewhat more direct computation of the desired form of the equations, in particular with the help of computer algebra. We actually used the two methods in order to check our results.

The quantities $\Phi^{(n)}$, $n > 2$, defined above arise by separating integrals from the remaining terms in the respective ncKP hierarchy equation. They show a certain (imperfect) building law which is related to the existence of recursion operators for the KP hierarchy, see appendix F.

With the help of (5.3), we can also express (5.1) in the form

$$\phi_{\theta_{m,n}} = \frac{1}{2}\left(\sum_{i=1}^{n+1} \sigma_i^{(m)} * \operatorname{res}(L^{n-i}) - \sum_{i=1}^{m+1} \sigma_i^{(n)} * \operatorname{res}(L^{m-i})\right)$$
$$= \frac{1}{2}\left(\sigma_{n+1}^{(m)} - \sigma_{m+1}^{(n)} - \sum_{i=1}^{n-1} \sigma_i^{(m)} * \sigma_1^{(n-i)} + \sum_{i=1}^{m-1} \sigma_i^{(n)} * \sigma_1^{(m-i)}\right) \qquad (6.9)$$

and use lemma 5.1 and appendix D to obtain expressions in terms of the u_k. Then we use again formulae from appendix A to find

$$\phi_{\theta_{1,2}} = \tfrac{1}{8}(\Phi^{(3)} - \phi_{xxx}) - \tfrac{3}{4}\phi_x^{*2} \qquad (6.10)$$

$$\phi_{\theta_{1,3}} = \tfrac{1}{8}\Phi^{(4)} - \tfrac{1}{8}\phi_{xxy} - \tfrac{1}{2}\{\phi_x, \phi_y\}_* + \tfrac{1}{4}[\phi_{xx}, \phi_x]_* \qquad (6.11)$$

$$\phi_{\theta_{1,4}} = \tfrac{1}{32}\big(3\Phi^{(5)} - 10\{\phi_x, \phi_{xxx} + \Phi^{(3)}\}_* - \phi_{xxxxx} - 2\phi_{xyy} - 10[\phi_y, \phi_{xx}]_*$$
$$+ 6[\phi_{xy}, \phi_x]_* - 10\phi_{xx}^{*2} - 4\phi_y^{*2} - 40\phi_x^{*3}\big) \qquad (6.12)$$

$$\phi_{\theta_{2,3}} = \tfrac{1}{32}\big(\Phi^{(5)} + 2\{\phi_x, \Phi^{(3)}\}_* + \phi_{xxxxx} - 2\phi_{xyy} + 10\big(\{\phi_x, \phi_{xxx}\}_* + \phi_{xx}^{*2}\big)$$
$$+ 2[\phi_y, \phi_{xx}]_* + 10[\phi_{xy}, \phi_x]_* - 20\phi_y^{*2} + 40\phi_x^{*3}\big). \qquad (6.13)$$

In accordance with (5.30), see also the last remark in section 5, we observe the following structure,

$$\mathrm{ncKP}^{(n)} = \Phi^{(n)} \qquad \phi_{\theta_{m,n}} = a_{m,n}\Phi^{(m+n)} + \cdots$$

where the first equation stands for the nth ncKP equation (with 'time' t_n), $a_{m,n}$ are constants, and the dots represent terms which are local in ϕ, $\Phi^{(k)}$, $k < m + n$. Probably, the recursion operators found in [22], see also appendix F, can be modified or supplemented by further recursion operators to cover the whole ncKP and also the xncKP hierarchy[18]. They should relate expressions containing the $\Phi^{(n)}$. We already followed another route towards explicit expressions for the xncKP hierarchy equations in section 5, which does not attempt to express the xncKP equations solely in terms of ϕ and its x- and y-derivatives, as well as x-integrals. In not insisting on eliminating derivatives with respect to the other variables t_n, we avoid integrals and often achieve simpler formulae (see (5.45), however). For example, the system composed of the potential ncKP equation (6.6) and the deformation equation (6.10) can be replaced by the considerably simpler one consisting of (5.40) and

$$\big(2\phi_{\theta_{1,2}} + \phi_{t_3}\big)_x - \phi_{yy} + 2[\phi_x, \phi_y]_* = 0. \qquad (6.14)$$

7. Reductions of the xncKP hierarchy

Let $\mathcal{A}$ be the algebra of pseudo-differential operators with coefficients which are differential polynomials in $\{u_i\}_{i=2}^{\infty}$. Let $\mathcal{I}_Q$ be the (two-sided) differential ideal generated by $Q \in \mathcal{A}$. The xncKP equations admit a *reduction* to $\mathcal{A}_Q = \mathcal{A}/\mathcal{I}_Q$ if

$$(\mathcal{I}_Q)_{t_n} \subset \mathcal{I}_Q \qquad (\mathcal{I}_Q)_{\theta_{k,l}} \subset \mathcal{I}_Q \qquad \forall n, k, l \in \mathbb{N} \qquad (7.1)$$

[18] See also [22–24], for example, for recursion operators of the KP hierarchy.

(see also [8], section 5.2). For an equality up to addition of terms lying in $\mathcal{I}_Q$ we write 'mod Q'. A special reduction is obtained by setting $Q = \bar{L}^{(N)}$ for some fixed $N \in \mathbb{N}$, so that $L^N = L^{(N)}$ mod Q. This is called *N-reduction* [2, 5, 9, 19, 25, 26] and reduces the KP hierarchy to the Gelfand–Dickey hierarchies (see [8], for example). Indeed,

$$(\bar{L}^{(N)})_{t_n} = ((L^N)_{t_n})_{<0} = ([L^{(n)}, L^N]_*)_{<0} = ([L^{(n)}, \bar{L}^{(N)}]_*)_{<0} \tag{7.2}$$

shows that the first of conditions (7.1) is satisfied [8]. Since $L^{(rN)} = (L^{(N)})^r = L^{rN}$ mod Q for all $r \in \mathbb{N}$, we have $L_{t_{rN}} = 0$ mod Q [8]. The extended hierarchy obtained by N-reduction still contains evolution equations in those variables t_n, for which N does not divide n. Writing

$$L^{(N)} = \partial^N + v_{N-2}\partial^{N-2} + v_{N-3}\partial^{N-3} + \cdots + v_0 \tag{7.3}$$

with (matrices of) functions v_i, the N-reduction constraint allows us to express the u_k, $k = 2, 3, \ldots$, as differential polynomials of v_i, $i = N-k, \ldots, N-2$ [8]. The ncKP equations are thus reduced to

$$(L^{(N)})_{t_n} = [L^{(n)}, L^{(N)}]_* \quad \text{mod } Q \qquad \forall n \in \mathbb{N}, \quad n/N \notin \mathbb{N}. \tag{7.4}$$

The relation[19]

$$(\bar{L}^{(N)})_{\theta_{m,n}} = \tfrac{1}{2}(L^{(n)} * \bar{L}^{(N)} * L^{(m)} - L^{(m)} * \bar{L}^{(N)} * L^{(n)} + \bar{L}^{(N)} * [L^{(m)}, L^{(n)}]_*)_{<0} + ([W^{(m,n)}, \bar{L}^{(N)}]_*)_{<0} \tag{7.5}$$

shows that also the second part of (7.1) holds. Furthermore, we find $L_{\theta_{kN,lN}} = 0$ mod Q for all $k, l \in \mathbb{N}$, and

$$L_{\theta_{kN,lN+r}} = \tfrac{1}{2} L_{t_{(k+l)N+r}} \quad \text{mod } Q \tag{7.6}$$

for all $k \in \mathbb{N}$, $l \in \mathbb{N} \cup \{0\}$, and $r = 1, 2, \ldots, N-1$.[20] Hence, after reduction, the fields only depend on the combination of variables $t_{(k+l)N+r} + \theta_{kN,lN+r}/2$.

As a consequence, N-reduction of xncKP hierarchy equations amounts to the following recipe. ϕ is only allowed to depend on t_{kN+r} and $\theta_{kN+r,lN+s}$, where $k, l = 0, 1, 2, \ldots$ and $r, s = 1, 2, \ldots, N-1$. Derivatives of ϕ with respect to t_{mN}, $m \in \mathbb{N}$, have to be dropped. Furthermore, each derivative with respect to a variable $\theta_{kN,lN+r}$ has to be replaced by $1/2$ times the derivative with respect to $t_{(k+l)N+r}$. In this way, the equations of the reduced hierarchy, expressed in terms of ϕ, are easily obtained from the formulae derived in the previous two sections.

Two examples of N-reductions are treated in the following subsections.

7.1. xncKdV hierarchy

The 2-reduction condition

$$L^2 = \partial^2 + u \tag{7.7}$$

leads to expressions for the variables u_k of the ncKP hierarchy in terms of the new variable u (see appendix E). The ncKdV hierarchy is the set of equations

$$u_{t_{2n+1}} = (L^2)_{t_{2n+1}} = [L^{(2n+1)}, L^2]_* \qquad n = 1, 2, \ldots. \tag{7.8}$$

[19] This expression is obtained by taking the negative part of (4.24) in [3], and using (7.2).

[20] The fact that not all of the θ-equations are independent from the reduced ncKP equations already follows from the following argument. Since ϕ does not depend on t_{kN}, there is no deformation of the product with respect to the parameters $\theta_{kN,lN+r}$ (see (1.3)). If we drop the dependence on all the variables $\theta_{kN+r,lN+s}$ and also the associated θ-equations, the $*$-product reduces to the ordinary one. Since ϕ is allowed to depend on the variables $\theta_{kN,lN+r}$, we still have non-trivial θ-equations in this case, the flows of which commute with those of the *classical* reduced KP hierarchy. If the latter hierarchy is already complete, it follows that these θ-equations must be equivalent to (combinations of the) reduced KP hierarchy equations.

According to (7.6),

$$\tfrac{1}{2}u_{t_{2(k+l)+1}} = u_{\theta_{2k,2l+1}} = \mathrm{res}(\bar{L}^{(2l+1)} * L^{(2k)})_x = \mathrm{res}(L^{2(k+l)+1})_x. \tag{7.9}$$

In fact, it is well known that the KdV hierarchy can be written in this way [27, 28]. According to Lax (see footnote 3 in [29]), this form of the KdV hierarchy has first been discovered by Gardner. Indeed, using (1.1), which implies $(L^m)_{t_n} = [L^{(n)}, L^m]_*$ for $m, n \in \mathbb{N}$, we find

$$\begin{aligned}\mathrm{res}(L^{2l+1})_{t_{2k+1}} &= \mathrm{res}(L^{2l+1})_{t_{2k+1}}\\ &= \mathrm{res}(L^{(2k+1)} * L^{2l+1} - L^{2l+1} * L^{(2k+1)})\\ &= \mathrm{res}((L^{2k+1} - \bar{L}^{(2k+1)}) * L^{2l+1} - \bar{L}^{(2l+1)} * L^{(2k+1)})\\ &= \mathrm{res}(L^{2(k+l+1)} - \bar{L}^{(2k+1)} * L^{(2l+1)} - \bar{L}^{(2l+1)} * L^{(2k+1)})\\ &= -\mathrm{res}(\bar{L}^{(2k+1)} * L^{(2l+1)} + \bar{L}^{(2l+1)} * L^{(2k+1)}).\end{aligned} \tag{7.10}$$

Since the right-hand side is symmetric in k and l, this implies that

$$\mathrm{res}(L^{2l+1})_{t_{2k+1}} = \mathrm{res}(L^{2k+1})_{t_{2l+1}} \tag{7.11}$$

which includes (7.9) as a special case (via $l \to 0$ and $k \to k + l$). The last equation implies the commutativity of the ncKdV flows.

The remaining deformation equations are given by

$$u_{\theta_{2k+1,2l+1}} = \mathrm{res}(\bar{L}^{(2l+1)} * L^{(2k+1)} - \bar{L}^{(2k+1)} * L^{(2l+1)})_x \tag{7.12}$$

Instead of evaluating (7.8) and (7.12) (see appendix E), we can apply more directly the simple reduction recipe, mentioned in the beginning of this section, to the equations of the xncKP hierarchy expressed in terms of the potential ϕ. Note that (1.8) and (7.7) imply

$$u = 2\phi_x. \tag{7.13}$$

Using $\phi_{\theta_{1,2}} = -\phi_{t_3}/2$ and $\phi_{\theta_{1,4}} = -\phi_{t_5}/2$, the deformation equations (5.40) and (5.42) reproduce the first two (potential) ncKdV equations:

$$\phi_{t_3} = \tfrac{1}{4}\phi_{xxx} + \tfrac{3}{2}\phi_x * \phi_x \tag{7.14}$$

$$\phi_{t_5} = \tfrac{1}{16}\phi_{xxxxx} + \tfrac{5}{8}\{\phi_x, \phi_{xxx}\}_* + \tfrac{5}{8}\phi_{xx}^{*2} + \tfrac{5}{2}\phi_x^{*3}. \tag{7.15}$$

Alternatively, the last equation is also obtained from (5.49) by using $\phi_{\theta_{2,3}} = -\phi_{t_5}/2$. Furthermore, from (5.41) with $\phi_y = 0 = \phi_{t_4}$ we recover (E.8), i.e.,

$$\phi_{\theta_{1,3}} = -\tfrac{1}{4}[\phi_x, \phi_{xx}]_*. \tag{7.16}$$

7.2. xncBoussinesq hierarchy

The *xncBoussinesq hierarchy* is obtained from the xncKP hierarchy by imposing the 3-reduction constraint

$$L^3 = \partial^3 + u\partial + v \tag{7.17}$$

with variables u and v. This leads to

$$\begin{aligned}&u_2 = \tfrac{1}{3}u \qquad u_3 = \tfrac{1}{3}(v - u_x) \qquad u_4 = \tfrac{1}{9}(2u_{xx} - u^2 - 3v_x)\\ &u_5 = \tfrac{1}{27}(6v_{xx} - 3v * u - 3u * v - 3u_{xxx} + 7u * u_x + 5u_x * u)\\ &u_6 = \tfrac{1}{81}(3u_{xxxx} - 9v_{xxx} - 15u_{xx} * u - 30u * u_{xx} + 21u * v_x + 15(v_x * u + u_x * v)\\ &\qquad + 30v * u_x - 45u_x * u_x - 9v^{*2} + 5u^{*3}).\end{aligned} \tag{7.18}$$

The equations of the ncBoussinesq hierarchy are given by

$$u_{t_n}\partial + v_{t_n} = (L^3)_{t_n} = [L^{(n)}, L^3]_* \qquad n = 2, 4, 5, 7, 8, \ldots. \tag{7.19}$$

For $n = 2$ this yields

$$u_y = 2v_x - u_{xx} \qquad v_y = v_{xx} - \tfrac{2}{3}(u_{xxx} + u * u_x - [u, v]_*) \tag{7.20}$$

where $y = t_2$. Introducing the potential ϕ via (1.8), we have

$$u = 3\phi_x \tag{7.21}$$

and the first equation leads to

$$v = \tfrac{3}{2}(\phi_y + \phi_{xx}) \tag{7.22}$$

which, inserted in the second equation, yields the (potential) *ncBoussinesq equation*

$$\phi_{yy} = -\tfrac{1}{3}\phi_{xxxx} - 2\big(\phi_x^{*2}\big)_x - 2[\phi_y, \phi_x]_*. \tag{7.23}$$

For $n = 4$, we find

$$u_{t_4} = \tfrac{2}{3}v_{xx} - \phi_{xxxx} + 2\{\phi_x, v\}_* - 3\big(\phi_x^{*2}\big)_x + [\phi_x, \phi_{xx}]_*. \tag{7.24}$$

With the above expression for v, this becomes

$$\phi_{t_4} = \tfrac{1}{3}(\phi_{xxy} + 3\{\phi_x, \phi_y\}_* + [\phi_x, \phi_{xx}]_*). \tag{7.25}$$

This equation is also obtained from (5.41) with the help of $\phi_{\theta_{1,3}} = -\phi_{t_4}/2$. Moreover, using $\phi_{\theta_{2,3}} = -\phi_{t_5}/2$ in (5.49), we obtain

$$\phi_{t_5} = \tfrac{5}{24}\phi_{xyy} - \tfrac{1}{24}\phi_{xxxxx} + \tfrac{5}{4}\phi_y^{*2} + \tfrac{5}{12}[\phi_x, \phi_{xy}]_* - \tfrac{5}{12}\{\phi_x, \phi_{xxx}\}_* - \tfrac{5}{12}\phi_{xx}^{*2} - \tfrac{5}{3}\phi_x^{*3}. \tag{7.26}$$

Furthermore, setting $\phi_{t_3} = 0$ in (5.40) and (5.42), leads to

$$\phi_{\theta_{1,2}} = -\tfrac{1}{6}\phi_{xxx} - \phi_x * \phi_x \tag{7.27}$$

$$\phi_{\theta_{1,4}} = \tfrac{3}{10}\phi_{t_5} - \tfrac{1}{8}\phi_{xyy} - \tfrac{1}{120}\phi_{xxxxx} - \tfrac{1}{12}\{\phi_x, \phi_{xxx}\}_* - \tfrac{1}{4}[\phi_x, \phi_{xy}]_* + \tfrac{1}{4}[\phi_{xx}, \phi_y]_* - \tfrac{1}{2}\phi_y^{*2}. \tag{7.28}$$

Equation (5.43) in this way reproduces the ncBoussinesq equation (7.23).

8. *N*-soliton solutions of some xncKP equations

In this section, we present *N*-soliton solutions of some of the xncKP equations, following [13]. We start with the ncKP equation and the first deformation equation (5.40). It is simpler, however, and equivalent to consider (6.14) instead of the ncKP equation. Inserting the formal series (see also [30])

$$\phi = \sum_{n=1}^{\infty} \epsilon^n \phi^{(n)} \tag{8.1}$$

in a parameter ϵ in both equations, and demanding that the resulting equations are satisfied order by order in ϵ, leads to

$$\phi^{(n)}_{\theta_{1,2}} - \frac{1}{6}\big(\phi^{(n)}_{t_3} - \phi^{(n)}_{xxx}\big) = -\sum_{r=1}^{n-1} \phi_x^{(r)} * \phi_x^{(n-r)} \tag{8.2}$$

$$\big(2\phi^{(n)}_{\theta_{1,2}} + \phi^{(n)}_{t_3}\big)_x - \phi^{(n)}_{yy} = -2\sum_{r=1}^{n-1}\big(\phi_x^{(r)} * \phi_y^{(n-r)} - \phi_y^{(r)} * \phi_x^{(n-r)}\big). \tag{8.3}$$

For $n = 1$, these are linear homogeneous equations which are solved by

$$\phi^{(1)} = \sum_{k=1}^{N} \phi_k \qquad \phi_k = c_k \, \mathrm{e}^{\xi(t,p_k)} * \mathrm{e}^{-\xi(t,q_k)} \tag{8.4}$$

where $N \in \mathbb{N}, \xi(t, p_k) = \sum_{r\geqslant 1} t_r p_k^r$, and c_k, p_k, q_k are constants. A solution of the inhomogeneous $n = 2$ equations is given by

$$\phi^{(2)} = \sum_{k,l=1}^{N} \frac{\phi_k * \phi_l}{q_k - p_l} \tag{8.5}$$

assuming $p_l \neq q_k$ for all $k, l = 1, \ldots, N$. A corresponding solution exists for arbitrary $n \in \mathbb{N}$. Indeed, introducing

$$\Phi^{(m,n)} = \frac{\phi_m * \phi_{m+1} * \cdots * \phi_n}{(q_m - p_{m+1})(q_{m+1} - p_{m+2}) \cdots (q_{n-1} - p_n)} \qquad m < n \tag{8.6}$$

and $\Phi^{(m,m)} = \phi_m$, we find that the equations

$$\Phi^{(1,n)}_{\theta_{1,2}} - \frac{1}{6}\big(\Phi^{(1,n)}_{t_3} - \Phi^{(1,n)}_{xxx}\big) = -\sum_{r=1}^{n-1} \Phi^{(1,r)}_x * \Phi^{(r+1,n)}_x \tag{8.7}$$

$$\big(2\Phi^{(1,n)}_{\theta_{1,2}} + \Phi^{(1,n)}_{t_3}\big)_x - \Phi^{(1,n)}_{yy} = -2\sum_{r=1}^{n-1} \big(\Phi^{(1,r)}_x * \Phi^{(r+1,n)}_y - \Phi^{(1,r)}_y * \Phi^{(r+1,n)}_x\big) \tag{8.8}$$

(cf (8.2) and (8.3)) are satisfied as a consequence of the algebraic identities

$$\Theta^{(1,n)}_{1,2} - \frac{1}{6}\big(T^{(1,n)}_3 - \big(T^{(1,n)}_1\big)^3\big) = -\sum_{r=1}^{n-1}(q_r - p_{r+1}) T^{(1,r)}_1 T^{(r+1,n)}_1 \tag{8.9}$$

$$\big(2\Theta^{(1,n)}_{1,2} + T^{(1,n)}_3\big) T^{(1,n)}_1 - \big(T^{(1,n)}_2\big)^2 = -2\sum_{r=1}^{n-1}(q_r - p_{r+1})\big(T^{(1,r)}_1 T^{(r+1,n)}_2 - T^{(1,r)}_2 T^{(r+1,n)}_1\big) \tag{8.10}$$

where we used the abbreviations

$$T^{(m,n)}_r = \sum_{k=m}^{n} \big(p_k^r - q_k^r\big) \tag{8.11}$$

$$\Theta^{(m,n)}_{r,s} = \frac{1}{2} \sum_{m\leqslant k<l\leqslant n} \big[\big(p_k^r - q_k^r\big)\big(p_l^s - q_l^s\big) - \big(p_k^s - q_k^s\big)\big(p_l^r - q_l^r\big)\big] - \frac{1}{2}\sum_{k=m}^{n} \big(p_k^r q_k^s - p_k^s q_k^r\big). \tag{8.12}$$

The first part of Θ is due to the interaction of different solitons (corresponding to terms $\phi_k * \phi_l$), while the second part is 'intrinsic', it originates from the $*$ in the definition of ϕ_k (see (8.4)). There is thus a formal analogy with orbital angular momentum and spin.

It follows that

$$\phi^{(n)} = \sum_{k_1,\ldots,k_n=1}^{N} \frac{\phi_{k_1} * \phi_{k_2} * \cdots * \phi_{k_n}}{\big(q_{k_1} - p_{k_2}\big)\big(q_{k_2} - p_{k_3}\big)\cdots\big(q_{k_{n-1}} - p_{k_n}\big)} \tag{8.13}$$

solves (8.2) and (8.3).

The fact that we were able to obtain solutions of (5.40) and (6.14) to *all* orders in ϵ is due to the existence of the identities (8.9) and (8.10) which hold for all $n \in \mathbb{N}$. Similar identities are associated with all other xncKP equations we have explored so far. For example, the above N-soliton solution of the ncKP equation also solves (5.41) as a consequence of the family of identities

$$\Theta_{1,3}^{(1,n)} = \frac{1}{4}\Bigg(T_4^{(1,n)} - \big(T_1^{(1,n)}\big)^2 T_2^{(1,n)} - 3\sum_{r=1}^{n-1}(q_r - p_{r+1})\big(T_1^{(1,r)}T_2^{(r+1,n)} + T_2^{(1,r)}T_1^{(r+1,n)}\big)$$
$$- \sum_{r=1}^{n-1}(q_r - p_{r+1})\big(T_1^{(1,r)}\big(T_1^{(r+1,n)}\big)^2 - \big(T_1^{(1,r)}\big)^2 T_1^{(r+1,n)}\big)\Bigg). \qquad (8.14)$$

The existence of such families of algebraic identities is the reason why certain partial differential equations can be solved via (8.1) universally to all orders. In principle, the argument could be reversed: finding a (suitable) family of identities, it should be possible to construct an associated partial differential equation which can be solved with the above method. We intend to address these questions in a separate work.

Remark. The N-soliton solution ϕ can be written in a more compact form. Using the bra–ket notation

$$\langle p| = (\tilde{c}_1\,\mathrm{e}^{\xi(\mathbf{x},p_1)}, \ldots, \tilde{c}_N\,\mathrm{e}^{\xi(\mathbf{x},p_N)}) \qquad |q\rangle = (\tilde{c}'_1\,\mathrm{e}^{-\xi(\mathbf{x},q_1)}, \ldots, \tilde{c}'_N\,\mathrm{e}^{-\xi(\mathbf{x},q_N)})^t \qquad (8.15)$$

with $\tilde{c}_k\,\tilde{c}'_k = c_k$ and introducing the $N \times N$ matrix

$$B = -\left(\frac{|q\rangle_k\langle p|_l}{q_k - p_l}\right) \qquad (8.16)$$

(where $|q\rangle_k$ is the kth component of $|q\rangle$) we obtain

$$\phi^{(n)} = (-1)^{n-1}\langle p| * B^{*(n-1)} * |q\rangle \qquad (8.17)$$

and, with the help of the geometric series formula, the following simple expression for the N-soliton solution:

$$\phi = \sum_{n=0}^{\infty}(-1)^n\langle p| * B^{*n} * |q\rangle = \langle p| * (I + B)^{*-1} * |q\rangle. \qquad (8.18)$$

In the commutative case with vanishing deformation, this can be rewritten as

$$\phi = \mathrm{Tr}((I + B)^{-1}B_x) = (\mathrm{Tr}\ln(I + B))_x = (\ln\tau)_x \qquad (8.19)$$

leading to Hirota's function $\tau = \det(I + B)$. In the presence of deformation, there seems to be no analogue of the τ-function.

Remark. Setting $q_k = -p_k$, we obtain $T_{2r}^{(m,n)} = 0$, $\Theta_{2r,2s}^{(m,n)} = 0$, and $\Theta_{2r,2s+1}^{(m,n)} = \frac{1}{2}T_{2(r+s)+1}^{(m,n)}$, in accordance with the 2-reduction conditions, see section 7. Hence we obtain N-soliton solutions of the xncKdV equations in this way. To find corresponding soliton solutions of the xncBoussinesq hierarchy requires us to set $q_k = \zeta p_k$ with ζ a primitive third root of unity.

9. Conclusions

The importance of the KP hierarchy in physics and many branches of mathematics is expected to be shared to a considerable extent by its deformations and extensions, thus in particular by the xncKP hierarchy. We have therefore started a thorough exploration of the latter.

We generalized a considerable part of the Sato formalism from the commutative KP case to the noncommutative deformed setting. Indeed, several important results, such as the well-known bilinear residue identities, extend to the xncKP hierarchy. A noncommutative analogue of Hirota's τ-function was not obtained in the present framework[21]. Since the τ-function is essential in further important developments in Sato theory, like the vertex operator formalism and the realization of the space of solutions of the ncKP hierarchy as an infinite-dimensional Grassmannian [4, 31], we were not able to achieve corresponding results for the deformed hierarchy. This still leaves the interesting question unanswered whether there is a certain 'noncommutative' analogue of the Grassmannian in the case of the noncommutative deformed hierarchy.

The KP hierarchy admits additional symmetries (see [8], for example) which commute with the hierarchy equations, but not among themselves. These symmetries depend explicitly on the evolution parameters t_n and are therefore non-autonomous (a property shared by our θ-equations). We have not explored whether these symmetries extend to symmetries of the extended deformed KP hierarchy. There are many more interesting properties of the KP hierarchy which may be carried over, but this needs further elaboration.

The usual definition of the KP hierarchy and its various generalizations by formulae such as (1.1) contains the equations in a rather implicit way. Quite involved computations are needed to derive explicit expressions for the members of the hierarchy. The computational expense is even higher in the case of the deformation equations. In section 5 we derived formulae which greatly facilitate the computation of the xncKP equations. In particular, via calculations in the xncKP hierarchy framework, we obtained corresponding formulae for the ncKP equations, which then hold for an arbitrary noncommutative associative product $*$. In this way, information about the classical (nc)KP hierarchy is obtained via an intermediate step into its Moyal-deformation and extension.

We also considered N-reductions of the xncKP hierarchy. In particular, the corresponding discussion in section 7 demonstrated that one can even learn something about *classical* sub-hierarchies from consideration of the xncKP system. This should provide some motivation to study more of the various facets of reductions in this framework. We certainly only touched upon this subject. In particular, there are generalizations of N-reductions (see [20, 26, 32–35], for example, and the references therein), which can be further generalized to our framework.

N-soliton solutions of the ncKP equation were shown to be also solutions of the first two deformation equations and it is likely that this holds in general. The emergence of families of algebraic identities in this connection seems to be an interesting route for further investigations on its own.

Some computations leading to results presented in this work have been carried out or checked with the help of the computer algebra software FORM [36–38].

Results similar to those reported in this work should be expected for deformations of the modified and discrete KP hierarchies [8] and generalizations (see [39], for example).

Appendix A. Formulae for the coefficients of L

In this appendix, we derive expressions for the coefficients u_k in terms of x- and y-derivatives of ϕ. The appearance of x-integrals in these expressions is the price one has to pay for the elimination of other t_n-derivatives. The second of equations (1.1) reads

$$L_y = [L^{(2)}, L] = [\partial^2 + 2u_2, L] = L_{xx} + 2L_x\partial - 2u_{2,x} + 2[u_2, \bar{L}]_* \tag{A.1}$$

[21] Corresponding progress, using an operator formalism, has been reported in [10].

where $y = t_2$. Inserting the series (1.2) leads to

$$\sum_{n=1}^{\infty} u_{n+1,y}\partial^{-n} = \sum_{n=1}^{\infty}(u_{n+1,xx} + 2u_{n+2,x} + 2u_2 * u_{n+1})\partial^{-n} \\ - 2\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k \binom{m+k-1}{k} u_{m+1} * \frac{\partial^k u_2}{\partial x^k}\partial^{-m-k} \tag{A.2}$$

with the help of the basic identity

$$\partial^{-m} f = \sum_{r=0}^{\infty} \binom{-m}{k} \frac{\partial^k f}{\partial x^k}\partial^{-m-k} = \sum_{k=0}^{\infty}(-1)^k \binom{m+k-1}{k} \frac{\partial^k f}{\partial x^k}\partial^{-m-k} \qquad m > 0 \tag{A.3}$$

for a function f. Hence

$$u_{n+1,y} = u_{n+1,xx} + 2u_{n+2,x} + 2[u_2, u_{n+1}]_* - 2\sum_{k=1}^{n-1}(-1)^k \binom{n-1}{k} u_{n-k+1} * \frac{\partial^k u_2}{\partial x^k} \tag{A.4}$$

which yields $u_{3,x} = (u_{2,y} - u_{2,xx})/2$ and

$$u_{n+1,x} = \frac{1}{2}(u_{n,y} - u_{n,xx}) - [u_2, u_n]_* + \sum_{k=1}^{n-2}(-1)^k \binom{n-2}{k} u_{n-k} * \frac{\partial^k u_2}{\partial x^k} \tag{A.5}$$

for $n > 2$. In particular,

$$\begin{aligned} u_{3,x} &= \tfrac{1}{2}(u_{2,y} - u_{2,xx}) \\ u_{4,x} &= \tfrac{1}{2}(u_{3,y} - u_{3,xx} - 2u_2 * u_{2,x} - 2[u_2, u_3]_*) \\ u_{5,x} &= \tfrac{1}{2}(u_{4,y} - u_{4,xx} - 4u_3 * u_{2,x} + 2u_2 * u_{2,xx} - 2[u_2, u_4]_*) \\ u_{6,x} &= \tfrac{1}{2}(u_{5,y} - u_{5,xx} - 6u_4 * u_{2,x} + 6u_3 * u_{2,xx} - 2u_2 * u_{2,xxx} - 2[u_2, u_5]_*) \\ u_{7,x} &= \tfrac{1}{2}(u_{6,y} - u_{6,xx} - 8u_5 * u_{2,x} + 12u_4 * u_{2,xx} - 8u_3 * u_{2,xxx} \\ &\quad + 2u_2 * u_{2,xxxx} - 2[u_2, u_6]_*). \end{aligned}$$

Introducing the potential ϕ via $u_2 = \phi_x$ leads to

$$u_3 = \tfrac{1}{2}(\phi_y - \phi_{xx}) \tag{A.6}$$

$$u_4 = \tfrac{1}{4}\left(\Phi^{(3)} + \phi_{xxx} - 2\phi_{xy} - 2\phi_x^{*2}\right) \tag{A.7}$$

$$u_5 = \tfrac{1}{8}(\Phi^{(4)} - 3\phi_{yy} - \phi_{xxxx} + 3\phi_{xxy} + 2\phi_x * \phi_y - 10\phi_y * \phi_x + 8\phi_x * \phi_{xx} + 4\phi_{xx} * \phi_x) \tag{A.8}$$

$$u_6 = \tfrac{1}{8}\left(\tfrac{1}{2}\Phi^{(5)} - 2\Phi^{(3)}_y + \phi_x * \Phi^{(3)} - 7\Phi^{(3)} * \phi_x + 8\phi_y * \phi_{xx} - 2\phi_y^{*2} - 3\phi_{xxx} * \phi_x + 4\phi_x^{*3} \right. \\ \left. + \left(3\phi_{yy} + \tfrac{1}{2}\phi_{xxxx} - 2\phi_{xxy} + 9\phi_y * \phi_x - \phi_x * \phi_y - 11\phi_x * \phi_{xx}\right)_x\right) \tag{A.9}$$

with the $\Phi^{(n)}$ defined in (6.2)–(6.5).

Appendix B. Residues of powers of *L*

In this appendix the residues of the first six powers of L are listed:

$$\begin{aligned} \mathrm{res}(L) &= u_2 \\ \mathrm{res}(L^2) &= 2u_3 + u_{2,x} \end{aligned}$$

$$\begin{aligned}
\mathrm{res}(L^3) &= 3u_4 + 3u_{3,x} + u_{2,xx} + 3u_2^{*2} \\
\mathrm{res}(L^4) &= 4u_5 + 6u_{4,x} + 4u_{3,xx} + u_{2,xxx} + 6(u_3 * u_2 + u_2 * u_3) + 4u_{2,x} * u_2 + 2u_2 * u_{2,x} \\
\mathrm{res}(L^5) &= 5u_6 + 10u_{5,x} + 10u_{4,xx} + 5u_{3,xxx} + u_{2,xxxx} + 10(u_4 * u_2 + u_2 * u_4) \\
&\quad + 10u_3^{*2} + 10u_{2,x} * u_3 + 10(u_{3,x} * u_2 + u_2 * u_{3,x}) + 10u_2^{*3} \\
&\quad + 5(u_{2,xx} * u_2 + u_2 * u_{2,xx} + u_{2,x} * u_{2,x}) \\
\mathrm{res}(L^6) &= 6u_7 + 15u_{6,x} + 20u_{5,xx} + 15u_{4,xxx} + 6u_{3,xxxx} + u_{2,xxxxx} + 15(u_5 * u_2 + u_2 * u_5) \\
&\quad + 20u_{4,x} * u_2 + 25u_2 * u_{4,x} + 15(u_4 * u_3 + u_3 * u_4) - 5u_4 * u_{2,x} \\
&\quad + 20u_{2,x} * u_4 + 15u_{3,xx} * u_2 + 20u_2 * u_{3,xx} + 20u_{3,x} * u_3 + 10u_3 * u_{3,x} \\
&\quad + 5u_{3,x} * u_{2,x} + 25u_{2,x} * u_{3,x} + 10u_3 * u_{2,xx} + 15u_{2,xx} * u_3 \\
&\quad + 20\big(u_3 * u_2^{*2} + u_2^{*2} * u_3\big) + 6u_{2,xxx} * u_2 + 4u_2 * u_{2,xxx} + 9u_{2,xx} * u_{2,x} \\
&\quad + 11u_{2,x} * u_{2,xx} + 15u_{2,x} * u_2^{*2} + 5u_2^{*2} * u_{2,x} + 20u_2 * u_3 * u_2 \\
&\quad + 10u_2 * u_{2,x} * u_2.
\end{aligned}$$

Note that $\mathrm{res}(L^n) = nu_{n+1} + \cdots$ where the remaining terms only involve the u_k with $k \leqslant n$. Hence, $u_{n+1} = \phi_{t_n}/n + \cdots$ by use of (1.9).

Appendix C. Evaluation of bilinear identities

Writing $X = \sum_{n=0}^{\infty} w_n \partial^{-n}$ with $w_0 = 1$ and $X^{-1} = \sum_{n=0}^{\infty} v_n \partial^{-n}$, we find the coefficients[22] v_n as differential polynomials in the w_n, using the basic formula (A.3):

$$\begin{aligned}
1 = X * X^{-1} &= \sum_{k,l=0}^{\infty} w_k \partial^{-k} * v_l \partial^{-l} = \sum_{k,l,r=0}^{\infty} \binom{-k}{r} w_k * \big(\partial_x^r v_l\big) \partial^{-k-l-r} \\
&= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} \sum_{r=0}^{m} \binom{-m+r}{r} w_{m-r} * \big(\partial_x^r v_{n-m}\big) \right] \partial^{-n}
\end{aligned} \tag{C.1}$$

where $\partial_x^r v_l = \partial^r v_l / \partial x^r$. Hence

$$\sum_{m=0}^{n} \sum_{r=0}^{m} \binom{-m+r}{r} w_{m-r} * \big(\partial_x^r v_{n-m}\big) = \delta_{n,0} \qquad n = 0, 1, \ldots \tag{C.2}$$

which leads to $v_0 = 1$, $v_1 = -w_1$, and

$$v_n = -w_n - \sum_{m=1}^{n-1} \sum_{r=0}^{m-1} (-1)^r \binom{m-1}{r} w_{m-r} * \big(\partial_x^r v_{n-m}\big) \qquad n = 2, 3, \ldots. \tag{C.3}$$

Furthermore, using $L = X * \partial * X^{-1} = \partial X * X^{-1} - X_x * X^{-1}$, we find

$$\begin{aligned}
\sum_{n=1}^{\infty} u_{n+1} \partial^{-n} &= L - \partial = -X_x * X^{-1} \\
&= -w_{1,x} \partial^{-1} - \sum_{n=2}^{\infty} \left[\sum_{m=1}^{n-1} \sum_{r=0}^{m-1} (-1)^r \binom{m-1}{r} w_{m-r,x} * \big(\partial_x^r v_{n-m}\big) + w_{n,x} \right] \partial^{-n}
\end{aligned} \tag{C.4}$$

[22] These coefficients are only used in this appendix. They are different from the v_n which appear in section 7.

from which we read off $u_2 = -w_{1,x}$ and

$$u_{n+1} = -w_{n,x} - \sum_{m=1}^{n-1}\sum_{r=0}^{m-1}(-1)^r \binom{m-1}{r} w_{m-r,x} * \left(\partial_x^r v_{n-m}\right) \qquad n = 2, 3, \ldots. \tag{C.5}$$

This determines the w_k in terms of the u_k. Setting $u_2 = \phi_x$, we find

$$w_1 = -\phi \qquad w_{2,x} = -u_3 + \phi_x * \phi \qquad w_{3,x} = -u_4 + u_3 * \phi - \phi_x * w_2 - \phi_x * \phi_x. \tag{C.6}$$

The Baker–Akhiezer function ψ is then determined via (2.5).

In order to elaborate the bilinear identities (4.4), we still need expressions in terms of the u_k for the coefficients $w_n^{(*)}$ introduced in (2.14). Using $X^\dagger = \sum_{n=0}^\infty (-\partial)^{-n} w_n^\dagger$ and $(X^\dagger)^{-1} = \sum_{n=0}^\infty \left(w_n^{(*)}\right)^\dagger (-\partial)^{-n}$ with $w_0^{(*)} = 1$, the coefficients $w_n^{(*)}$ are determined by

$$\begin{aligned} 1 = (X^\dagger)^{-1} * X^\dagger &= \sum_{k,l=0}^{\infty} \left(w_k^{(*)}\right)^\dagger (-\partial)^{-k-l} w_l^\dagger \\ &= \sum_{k,l,r=0}^{\infty} (-1)^{k+l} \binom{-k-l}{r} \left(w_k^{(*)}\right)^\dagger * \left(\partial_x^r w_l^\dagger\right) \partial^{-k-l-r} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{r=0}^{m} (-1)^{n-r} \binom{-n+r}{r} \left(w_{m-r}^{(*)}\right)^\dagger * \left(\partial_x^r w_{n-m}^\dagger\right) \partial^{-n}. \end{aligned} \tag{C.7}$$

Reading off the coefficients of powers of ∂ and applying the involution †, leads to

$$\sum_{m=0}^{n}\sum_{r=0}^{m} \binom{n-1}{r} \left(\partial_x^r w_{n-m}\right) * w_{m-r}^{(*)} = 0 \qquad n = 1, 2, \ldots \tag{C.8}$$

which implies $w_1^{(*)} = -w_1$ and

$$w_n^{(*)} = -w_n - \sum_{m=1}^{n-1}\sum_{r=0}^{m} \binom{n-1}{r} \left(\partial_x^r w_{n-m}\right) * w_{m-r}^{(*)} \qquad n = 2, 3, \ldots. \tag{C.9}$$

In particular,

$$w_2^{(*)} = -w_2 + w_1^{*2} - w_{1,x} \tag{C.10}$$

$$w_3^{(*)} = -w_3 + w_1 * w_2 + w_2 * w_1 - w_1^{*3} - 2w_{2,x} + w_1 * w_{1,x} + 2w_{1,x} * w_1 - w_{1,xx}. \tag{C.11}$$

In terms of $\hat{w}$ and $\hat{w}^*$, the bilinear identities take the form

$$\mathrm{res}_\lambda\left(\partial_{t_1}^{i_1} \cdots \partial_{t_p}^{i_p} \partial_{\theta_{1,2}}^{j_{1,2}} \cdots \partial_{\theta_{m,n}}^{j_{m,n}} (\hat{w} * \mathrm{e}^{\xi}) * \mathrm{e}^{-\xi} * \hat{w}^*\right) = 0. \tag{C.12}$$

Let us evaluate some of them. The simplest case is

$$\mathrm{res}_\lambda(\hat{w} * \hat{w}^*) = \sum_{k,l=0}^{\infty} \mathrm{res}_\lambda\left(w_k * w_l^{(*)} \lambda^{-k-l}\right) = w_1 + w_1^{(*)} \equiv 0. \tag{C.13}$$

Furthermore, using

$$\psi_{t_i} = \left(\hat{w}_{t_i} + \hat{w}\lambda^i\right) * \mathrm{e}^{\xi} \qquad \psi_{t_i t_j} = \left(\hat{w}_{t_i t_j} + \hat{w}_{t_i}\lambda^j + \hat{w}_{t_j}\lambda^i + \hat{w}\lambda^{i+j}\right) * \mathrm{e}^{\xi} \tag{C.14}$$

we obtain

$$\mathrm{res}_\lambda\left[(\hat{w}_{t_i} + \hat{w}\lambda^i) * \hat{w}^*\right] = 0 \tag{C.15}$$

$$\mathrm{res}_\lambda\big[(\hat{w}_{t_it_j} + \hat{w}_{t_i}\lambda^j + \hat{w}_{t_j}\lambda^i + \hat{w}\lambda^{i+j}) * \hat{w}^*\big] = 0 \tag{C.16}$$

and thus

$$0 = w_{1,t_i} + \sum_{k=0}^{i+1} w_k * w^{(*)}_{i-k+1} \tag{C.17}$$

$$0 = w_{1,t_it_j} + \sum_{k=1}^{i+1} w_{k,t_j} * w^{(*)}_{i-k+1} + \sum_{k=1}^{j+1} w_{k,t_i} * w^{(*)}_{j-k+1} + \sum_{k=0}^{i+j+1} w_k * w^{(*)}_{i+j-k+1}. \tag{C.18}$$

Of particular interest for us is the case where (C.12) only contains a single θ-derivative, since from it we recover the extension of the ncKP hierarchy. Using

$$\psi_{\theta_{i,j}} = \big(\hat{w}_{\theta_{i,j}} + \tfrac{1}{2}(\hat{w}_{t_i}\lambda^j - \hat{w}_{t_j}\lambda^i)\big) * \mathrm{e}^\xi \tag{C.19}$$

we obtain

$$\mathrm{res}_\lambda\big[\big(\hat{w}_{\theta_{i,j}} + \tfrac{1}{2}\big(\hat{w}_{t_i}\lambda^j - \hat{w}_{t_j}\lambda^i\big)\big) * \hat{w}^*\big] = 0 \tag{C.20}$$

and thus

$$w_{1,\theta_{i,j}} + \frac{1}{2}\sum_{k=1}^{j+1} w_{k,t_i} * w^{(*)}_{j-k+1} - \frac{1}{2}\sum_{k=1}^{i+1} w_{k,t_j} * w^{(*)}_{i-k+1} = 0 \tag{C.21}$$

which is another expression for the tower of ncKP deformation equations (5.1), respectively (6.9). For example, using $w^{(*)}_0 = 1$, $w^{(*)}_1 = -w_1$, and (C.6), (C.10), (A.6), (A.7), one recovers (6.10).

Appendix D. Computation of the σ-coefficients

Expressions for the coefficients $\sigma^{(1)}_m$ in terms of the u_k can be obtained from (5.3) with $n = 1$ as follows. Inserting $L^{(1)} = \partial$, this leads to

$$\sum_{j=1}^{\infty}\big(u_{j+1}\partial^{-j} + \sigma^{(1)}_j * L^{-j}\big) = 0 \tag{D.1}$$

and thus

$$\sigma^{(1)}_m = -u_{m+1} - \sum_{j=1}^{m-2}\sigma^{(1)}_j * (L^{-j})_{-m} \tag{D.2}$$

where the coefficients $(L^{-j})_{-m}$ can be expressed in terms of the coefficients ℓ_k of[23]

$$L^{-1} = \sum_{i=1}^{\infty}\ell_i\partial^{-i} \tag{D.3}$$

and their x-derivatives. One finds

$$\ell_1 = 1, \quad \ell_2 = 0, \quad \ell_3 = -u_2, \quad \ell_4 = -u_3 + u_{2,x}$$
$$\ell_{i+1} = -\sum_{k=0}^{i-2}(-1)^k\frac{\partial^k u_{i-k}}{\partial x^k} - \sum_{j=3}^{i-1}\ell_j * \sum_{k=0}^{i-j-1}(-1)^k\binom{j+k-1}{k}\frac{\partial^k u_{i-j-k+1}}{\partial x^k} \qquad i > 3 \tag{D.4}$$

[23] L has a unique left inverse as a formal series. L^{-1} is also a right inverse as a consequence of associativity.

so that

$$\ell_5 = -u_4 + u_{3,x} - u_{2,xx} + u_2^{*2}$$
$$\ell_6 = -u_5 + u_{4,x} - u_{3,xx} + u_{2,xxx} - 3u_2 * u_{2,x} - u_{2,x} * u_2 + u_2 * u_3 + u_3 * u_2$$
$$\begin{aligned}\ell_7 = -u_6 + u_{5,x} - u_{4,xx} + u_{3,xxx} - u_{2,xxxx} + 6u_2 * u_{2,xx} + u_{2,xx} * u_2 \\ - 3u_2 * u_{3,x} - u_{3,x} * u_2 + 4u_{2,x} * u_{2,x} - u_{2,x} * u_3 \\ - 4u_3 * u_{2,x} + u_3^{*2} - u_2^{*3} + u_2 * u_4 + u_4 * u_2\end{aligned}$$

and so forth. Furthermore, with the help of the identity (A.3) one obtains

$$L^{-j} = \sum_{\substack{i_1,\ldots,i_j=1 \\ k_1,\ldots,k_{j-1}=0}}^{\infty} (-1)^{\sum_{r=1}^{j-1} k_r} \ell_{i_1} \left(\prod_{s=1}^{j-1} \binom{\sum_{r=1}^{s}(i_r + k_r) - 1}{k_s} \frac{\partial^s}{\partial x^s} \ell_{i_{s+1}} \right) \partial^{-\sum_{r=1}^{j} i_r - \sum_{r=1}^{j-1} k_r} \tag{D.5}$$

for $j > 1$, which yields

$$\begin{aligned}(L^{-j})_{-m} = \sum_{\substack{i_1,\ldots,i_j \geqslant 1 \\ k_1,\ldots,k_{j-1} \geqslant 0 \\ i_1+\cdots i_j+k_1+\cdots+k_{j-1}=m}} (-1)^{\sum_{r=1}^{j-1} k_r} \binom{i_1 + k_1 - 1}{k_1} \binom{i_1 + i_2 + k_1 + k_2 - 1}{k_2} \\ \cdots \binom{i_1 + \cdots i_{j-1} + k_1 + \cdots + k_{j-1} - 1}{k_{j-1}} \ell_{i_1} \frac{\partial^{k_1} \ell_{i_2}}{\partial x^{k_1}} \cdots \frac{\partial^{k_{j-1}} \ell_{i_j}}{\partial x^{k_{j-1}}}.\end{aligned} \tag{D.6}$$

Inspection of this formula shows that $(L^{-j})_{-m} = 0$ if $m < j$, $(L^{-j})_{-j} = 1$, $(L^{-j})_{-j-1} = 0$, and $(L^{-j})_{-j-2} = j\ell_3 = -ju_2$. Hence $L^{-j} = \partial^{-j} - ju_2\partial^{-j-2} + \cdots$. Clearly, $(L^{-1})_{-m} = \ell_m$.

The formulae presented above are suitable for evaluation with computer algebra. In particular, one obtains

$$\sigma_2^{(1)} = -u_3$$
$$\sigma_3^{(1)} = -u_4 - u_2^{*2}$$
$$\sigma_4^{(1)} = -u_5 - 2u_3 * u_2 - u_2 * u_3 + u_2 * u_{2,x}$$
$$\sigma_5^{(1)} = -u_6 - 3u_4 * u_2 - u_2 * u_4 - 2u_3^{*2} + 3u_3 * u_{2,x} + u_2 * u_{3,x} - u_2 * u_{2,xx} - 2u_2^{*3}.$$

From (5.6) we find, for example,

$$\sigma_2^{(2)} = -2u_4 - u_{3,x} - u_2^{*2}$$
$$\sigma_3^{(2)} = -2u_5 - u_{4,x} - 3u_3 * u_2 - u_2 * u_3 - u_{2,x} * u_2 + u_2 * u_{2,x}$$
$$\begin{aligned}\sigma_4^{(2)} = -2u_6 - u_{5,x} - 5u_4 * u_2 - u_2 * u_4 - 2u_{3,x} * u_2 + u_2 * u_{3,x} - 3u_3^{*2} \\ + 4u_3 * u_{2,x} - u_{2,x} * u_3 + u_{2,x} * u_{2,x} - u_2 * u_{2,xx} - 2u_2^{*3}\end{aligned}$$
$$\sigma_2^{(3)} = -3u_5 - 3u_{4,x} - u_{3,xx} - 3(u_3 * u_2 + u_2 * u_3) + u_2 * u_{2,x} - u_{2,x} * u_2$$
$$\begin{aligned}\sigma_3^{(3)} = -3u_6 - 3u_{5,x} - u_{4,xx} - 6u_4 * u_2 - 3u_2 * u_4 - 4u_{3,x} * u_2 + u_2 * u_{3,x} - 3u_3^{*2} \\ + 4u_3 * u_{2,x} - u_{2,x} * u_3 - u_{2,xx} * u_2 - u_2 * u_{2,xx} + u_{2,x} * u_{2,x} - 4u_2^{*3}\end{aligned}$$
$$\begin{aligned}\sigma_2^{(4)} = -4u_6 - 6u_{5,x} - 4u_{4,xx} - u_{3,xxx} - 6(u_4 * u_2 + u_2 * u_4) - 4u_{3,x} * u_2 \\ - 2u_2 * u_{3,x} - 6u_3^{*2} + 4u_3 * u_{2,x} - 4u_{2,x} * u_3 - u_{2,xx} * u_2 \\ - u_2 * u_{2,xx} + u_{2,x} * u_{2,x} - 4u_2^{*3}.\end{aligned}$$

Note also that $\sigma_1^{(n)} = -\mathrm{res}(L^n)$ with corresponding expressions listed in appendix B. The results in this appendix in fact hold for an arbitrary associative product for which ∂ is a derivation. In the commutative case, corresponding expressions for the $\sigma_m^{(n)}$ can be found in the appendix of [19].

Appendix E. Some xncKdV formulae

In particular, (7.7) leads to

$$\begin{aligned}
&u_2 = 2^{-1}u \qquad u_3 = 2^{-2}(-u_x) \qquad u_4 = 2^{-3}(u_{xx} - u^{*2})\\
&u_5 = 2^{-4}(-u_{xxx} + 4u * u_x + 2u_x * u)\\
&u_6 = 2^{-5}(u_{xxxx} - 11u * u_{xx} - 3u_{xx} * u - 11u_x * u_x + 2u^{*3})\\
&u_7 = 2^{-6}(-u_{xxxxx} + 26u * u_{xxx} + 4u_{xxx} * u + 39u_x * u_{xx}\\
&\qquad + 21u_{xx} * u_x - 15u^{*2} * u_x - 10u * u_x * u - 5u_x * u^{*2})\\
&u_8 = 2^{-7}(u_{xxxxxx} - 5u_{xxxx} * u - 57u * u_{xxxx} - 34u_{xxx} * u_x - 114u_x * u_{xxx}\\
&\qquad - 91u_{xx}^{*2} + 9u_{xx} * u^{*2} + 69u^{*2} * u_{xx} + 32u * u_{xx} * u + 32u_x^{*2} * u\\
&\qquad + 92u * u_x^{*2} + 46u_x * u * u_x - 5u^{*4}).
\end{aligned} \tag{E.1}$$

By use of these expressions, we find

$$L^{(3)} = \partial^3 + \tfrac{3}{2}u\partial + \tfrac{3}{4}u_x \tag{E.2}$$

$$L^{(5)} = \partial^5 + \tfrac{5}{2}u\partial^3 + \tfrac{15}{4}u_x\partial^2 + \tfrac{5}{8}(5u_{xx} + 3u^{*2})\partial + \tfrac{5}{16}(3u_{xxx} + 2u_x * u + 4u * u_x) \tag{E.3}$$

$$\begin{aligned}
L^{(7)} = {}& \partial^7 + \tfrac{7}{2}u\partial^5 + \tfrac{35}{4}u_x\partial^4 + \tfrac{35}{8}(3u_{xx} + u^{*2})\partial^3 + \tfrac{5}{16}(35u_{xxx} + 14u_x * u + 28u * u_x)\partial^2\\
&+ \tfrac{1}{32}(161u_{xxxx} + 105u_{xx} * u + 245u_x^{*2} + 245u * u_{xx} + 70u^{*3})\partial\\
&+ \tfrac{1}{64}(63u_{xxxxx} + 56u_{xxx} * u + 189u_{xx} * u_x + 231u_x * u_{xx}\\
&+ 35u_x * u^{*2} + 154u * u_{xxx} + 70u * u_x * u + 105u^{*2} * u_x).
\end{aligned} \tag{E.4}$$

The first three non-trivial equations resulting from (7.8) are

$$u_{t_3} = 2^{-2}(u_{xx} + 3u^{*2})_x \tag{E.5}$$

$$u_{t_5} = 2^{-4}\left(u_{xxxx} + 5(u * u_{xx} + u_{xx} * u) + 5u_x^{*2} + 10u^{*3}\right)_x \tag{E.6}$$

$$\begin{aligned}
u_{t_7} = 2^{-6}\big(&u_{xxxxxx} + 7(u * u_{xxxx} + u_{xxxx} * u) + 14(u_x * u_{xxx} + u_{xxx} * u_x)\\
&+ 21u_{xx}^{*2} + 7(3u^{*2} * u_{xx} + 4u * u_{xx} * u + 3u_{xx} * u^{*2})\\
&+ 14(2u_x^{*2} * u + u_x * u * u_x + 2u * u_x^{*2}) + 35u^{*4}\big)_x
\end{aligned} \tag{E.7}$$

starting with the ncKdV equation, and from (7.12) we obtain

$$u_{\theta_{1,3}} = 2^{-3}([u_x, u]_*)_x \tag{E.8}$$

$$u_{\theta_{1,5}} = 2^{-5}([u_{xxx}, u]_* - [u_{xx}, u_x]_* + 5[u_x, u^{*2}]_*)_x \tag{E.9}$$

These equations have already been found in [3] by reduction of the ncAKNS hierarchy. They are recovered from (3.46)–(3.48) and (3.49)–(3.50) in [3] via $u \mapsto -u, t_{2n+1} \mapsto 2^{-2n}t_{2n+1}, \theta_{2n+1} \mapsto 2^{-2n}\theta_{1,2n+1}$. Furthermore,

$$\begin{aligned}
u_{\theta_{3,5}} = 2^{-7}(&[u_{xxx}, u_{xx}]_* + 3[u_{xxx}, u^{*2}]_* + 6[u_x, u^{*3}]_* + 4(u * u_x * u_{xx} - u_{xx} * u_x * u)\\
&+ 2(u_x * u * u_{xx} - u_{xx} * u * u_x) + 12(u * u_x * u^{*2} - u^{*2} * u_x * u))_x.
\end{aligned} \tag{E.10}$$

Appendix F. Dorfman–Fokas recursion operator

In this appendix, we recall some results from [22] and draw the relation with the present work. In the following, the operator D_x acts as differentiation with respect to x on the ring $\mathcal{R}[\partial_y]$ generated by ∂_y and ($N \times N$-matrices of) smooth functions of x and y. In particular, $D_x \partial_y = 0$. D_x^{-1} denotes the formal inverse of D_x (integration). Furthermore, we introduce the adjoint actions $\mathrm{ad}_V W = V * W - W * V$ and $\mathrm{ad}_V^+ W = V * W + W * V$. Using

$$\Psi = D_x^2 - D_x^{-1}\,\mathrm{ad}_V^+ D_x - \mathrm{ad}_V^+ + D_x^{-1}\,\mathrm{ad}_V D_x^{-1}\,\mathrm{ad}_V \tag{F.1}$$

where

$$V = -2u_2 + \partial_y = -2\phi_x + \partial_y \tag{F.2}$$

one defines recursively operators S_n (acting on $\mathcal{R}[\partial_y]$) via

$$S_0 = \Psi + 2R_{\partial_y} \qquad S_{n+1} = \Psi S_n + 4 S_n R_{\partial_y}. \tag{F.3}$$

Here R_{∂_y} is the operator of right multiplication by ∂_y. Acting on 1, the operators S_n produce functions, i.e., elements of $\mathcal{R}[\partial_y]$ which do not contain ∂_y. In [22], the series of equations

$$\phi_{t_{2n+1}} = 4^{-(n+1)} S_n 1 \qquad n = 0, 1, 2, \ldots \tag{F.4}$$

has been named *generalized KP hierarchy*. It corresponds, however, only to half of the (noncommutative) KP hierarchy, as made manifest by the notation used in (F.4). Although the ncKP equation (6.6) is easily recovered from the above formulae, the computation of higher odd ncKP equations turns out to be very time-consuming.

Note that, for example, $\Phi^{(3)} = D_x^{-1}\,\mathrm{ad}_V \Phi^{(2)} = \left(D_x^{-1}\,\mathrm{ad}_V\right)^2 \Phi^{(1)}$ with the $\Phi^{(n)}, n = 1, 2, 3$, defined in section 6. Furthermore, we have the following result.

Lemma.

$$\phi_{\theta_{1,n}} = \tfrac{1}{2}\left(D_x^{-1}\,\mathrm{ad}_V \phi_{t_n} - \phi_{t_{n+1}}\right) \qquad n = 2, 3, \ldots. \tag{F.5}$$

Proof. Equation (5.21) with $i = 2$ and $m = 1$ reads

$$\eta_{2,x}^{(n)} = \eta_{2,t_n}^{(1)} - \left[\phi_x, \phi_{t_n}\right]_* = \tfrac{1}{2}\phi_{yt_n} + \tfrac{1}{2}\phi_{xxt_n} - \left[\phi_x, \phi_{t_n}\right]_*$$

using (5.23). By application of the ω-involution (see section 5), this leads to

$$\sigma_{2,x}^{(n)} = -\tfrac{1}{2}\phi_{yt_n} + \tfrac{1}{2}\phi_{xxt_n} + \left[\phi_x, \phi_{t_n}\right]_*.$$

The difference of both equations is

$$\left(\eta_2^{(n)} - \sigma_2^{(n)}\right)_x = \phi_{yt_n} - 2\left[\phi_x, \phi_{t_n}\right]_*$$

and thus

$$\eta_2^{(n)} - \sigma_2^{(n)} = D^{-1}\,\mathrm{ad}_V \phi_{t_n}.$$

Inserting this expression in (5.32), we get (F.5). □

The proof of this lemma can be easily generalized to obtain a corresponding expression for $\phi_{\theta_{2,n}}$:

$$\begin{aligned}\phi_{\theta_{2,n}} &= -\tfrac{1}{2}\phi_{t_{n+2}} + \tfrac{1}{4}\phi_{t_n xx} + \left\{\phi_x, \phi_{t_n}\right\}_* - \tfrac{1}{2}D_x^{-1}\left\{\phi_{xx}, \phi_{t_n}\right\}_* + \tfrac{1}{4}\left(D_x^{-1}\,\mathrm{ad}_V\right)^2 \phi_{t_n} \\ &= -\tfrac{1}{2}\phi_{t_{n+2}} + \tfrac{1}{4}\Psi\phi_{t_n} + \tfrac{1}{2}\left\{\partial_y, \phi_{t_n}\right\}.\end{aligned} \tag{F.6}$$

References

[1] Kupershmidt B A 2000 KP or mKP *Mathematical Surveys and Monographs* vol 78 (Providence, RI: American Mathematical Society)

[2] Hamanaka M 2003 Commuting flows and conservation laws for noncommutative Lax hierarchies *Preprint* hep-th/0311206

[3] Dimakis A and Müller-Hoissen F 2004 Extension of noncommutative soliton hierarchies *J. Phys. A: Math. Gen.* **37** 4069–84

[4] Sato M and Sato Y 1982 Soliton equations as dynamical systems on infinite dimensional Grassmann manifold *Nonlinear Partial Differential Equations in Applied Science* (*Lecture Notes in Num. Appl. Anal.* vol 5) ed H Fujita, P D Lax and G Strang (Amsterdam: North-Holland) pp 259–71

[5] Date E, Kashiwara M, Jimbo M and Miwa T 1983 Transformation groups for soliton equations *Non-linear Integrable Systems—Classical Theory and Quantum Theory* ed M Jimbo and T Miwa (Singapore: World Scientific) pp 39–119

[6] Segal G and Wilson G 1985 Loop groups and equations of KdV type *Publ. Math. IHES* **61** 5–65

[7] Babelon O, Bernard D and Talon M 2003 *Introduction to Classical Integrable Systems* (Cambridge: Cambridge University Press)

[8] Dickey L A 2003 *Soliton Equations and Hamiltonian Systems* (Singapore: World Scientific)

[9] Ohta Y, Satsuma J, Takahashi D and Tokihiro T 1988 An elementary introduction to Sato theory *Prog. Theor. Phys. Suppl.* **94** 210–41

[10] Shigechi K, Wadati M and Wang N 2004 WDVV equation and triple-product relation *Preprint* hep-th/0404249

[11] Lechtenfeld O, Mazzanti L, Penati S, Popov A D and Tamassia L 2004 Integrable noncommutative sine-Gordon model *Preprint* hep-th/0406065

[12] Dimakis A and Müller-Hoissen F 2000 The Korteweg-de-Vries equation on a noncommutative space-time *Phys. Lett.* A **278** 139–45

[13] Paniak L D 2001 Exact noncommutative KP and KdV multi-solitons *Preprint* hep-th/0105185

[14] Wang N and Wadati M 2003 Noncommutative extension of $\bar{\partial}$-dressing method *J. Phys. Soc. Japan* **72** 1366–73

[15] Wang N and Wadati M 2003 Exact multi-soliton solutions of noncommutative KP equation *J. Phys. Soc. Japan* **72** 1881–8

[16] Wang N and Wadati M 2003 A new approach to noncommutative soliton equations *J. Phys. Soc. Japan* **72** 3055–62

[17] Mulase M 1984 Complete integrability of the Kadomtsev–Petviashvili equation *Adv. Math.* **54** 57–66

[18] Wilson G 1981 On two constructions of conservation laws for Lax equations *Q. J. Math. Oxford* **32** 491–512

[19] Matsukidaira J, Satsuma J and Strampp W 1990 Conserved quantities and symmetries of KP hierarchy *J. Math. Phys.* **31** 1426–34

[20] Aratyn H 1995 Integrable Lax hierarchies, their symmetry reductions and multi-matrix models *Preprint* hep-th/9503211

[21] Vladimirov A A 2004 Lectures on integrable hierarchies and vertex operators *Preprint* hep-th/0402097

[22] Dorfman I Ya and Fokas A S 1992 Hamiltonian theory over noncommutative rings and integrability in multidimensions *J. Math. Phys.* **33** 2504–14

[23] Oevel W and Fuchssteiner B 1982 Explicit formulas for symmetries and conservation laws of the Kadomtsev–Petviashvili equation *Phys. Lett.* A **88** 323–7

[24] Fokas A S and Santini P M 1986 The recursion operator of the Kadomtsev–Petviashvili equation and the squared eigenfunctions of the Schrödinger operator *Stud. Appl. Math.* **75** 179–86

[25] Strampp W and Oevel W 1990 Recursion operators and Hamiltonian structures in Sato's theory *Lett. Math. Phys.* **20** 195–210

[26] Konopelchenko B G and Oevel W 1993 An r-matrix approach to nonstandard classes of integrable equations *Publ. RIMS* **29** 581–666

[27] Avramidi I G and Schimming R 2000 A new explicit expression for the Korteweg–De Vries hierarchy *Math. Nachr.* **219** 45–64

[28] Arbarello E 2002 Sketches of KDV *Contemp. Math.* **312** 9–69

[29] Lax P D 1968 Integrals of nonlinear equations of evolution and solitary waves *Commun. Pure Appl. Math.* **21** 467–90

[30] Okhuma K and Wadati M 1983 The Kadomtsev–Petviashvili equation: the trace method and the soliton resonances *J. Phys. Soc. Japan* **52** 749–60

[31] Miwa T, Jimbo M and Date E 2000 *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras* (Cambridge: Cambridge University Press)

[32] Konopelchenko B G and Strampp W 1992 New reductions of the Kadomtsev–Petviashvili and two-dimensional Toda lattice hierarchies via symmetry constraints *J. Math. Phys.* **33** 3676–86

[33] Cheng Yi 1992 Constraints on the Kadomtsev–Petviashvili hierarchy *J. Math. Phys.* **33** 3774–82
[34] Cheng Yi 1992 Constraints of integrable systems: from higher to lower dimensions *Phys. Lett.* A **166** 217–23
[35] Loris I and Willox R 1997 KP symmetry reductions and a generalized constraint *J. Phys. A: Math. Gen.* **30** 6925–38
[36] Vermaseren J A M 2000 New features of FORM *Preprint* math-ph/0010025
[37] Vermaseren J A M 2002 *FORM Reference Manual* (Amsterdam: NIKHEF)
[38] Heck A 2000 *FORM for Pedestrians* (Amsterdam: NIKHEF)
[39] Takebe T 2002 A note on the modified KP hierarchy and its (yet another) dispersionless limit *Lett. Math. Phys.* **59** 157–72